The Scattering of Elastic Waves by Rough Surfaces

A thesis for the degree of Doctor of Philosophy

by

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Abstract

We consider the scattering of elastic waves by an unbounded surface on which the displacement vanishes. The wave field is assumed to be time-harmonic and the propagation medium to be homogeneous and isotropic. The scattering surface is assumed to be given as a graph of a bounded function $f \in C^{1,\alpha}$, but otherwise no assumptions are made.

The problem is formulated as a boundary value problem for the scattered field in the unbounded domain above the scattering surface. This boundary value problem formulation includes a novel radiation condition characterising upward propagating waves. The way in which this radiation condition generalises other radiation conditions commonly employed in elastic wave scattering problems is discussed in detail. It is then shown that the boundary value problem, and thus the scattering problem, admits at most one solution for a general class of incident fields including plane and cylindrical waves.

Existence of solution is established via the boundary integral equation method. The properties of elastic single- and double-layer potentials on rough surfaces are studied with an emphasis on obtaining estimates uniformly for classes of such surfaces. An equivalent formulation of the scattering problem as a boundary integral equation of the second kind is obtained. Since the scatterer is unbounded the integral operator in this equation is not compact, and nor is the equation of a standard singular type previously studied. Thus, a new solvability theory is developed, which establishes solvability of the integral equation in the space of bounded and continuous functions, and also in all L^p-spaces, $1 \leq p \leq \infty$.

Contents

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Chapter 1

Introduction

The problem of scattering of waves by objects with features of a size comparable to the wave-length has been of continuing interest to mathematicians for a long time. The roots of the theory were developed by Lord Rayleigh, A. Sommerfeld and others in the second half of the 19th century, but even today many questions remain unanswered.

A problem that has attracted considerable attention over the last decade by both mathematicians and engineers is the two-dimensional problem of scattering of a wave by an effectively unbounded surface with features of a dimension comparable to the wave-length. Mathematically such a surface is usually described as the graph of bounded function and it is termed a *rough surface*.

In the case of an incident acoustic or electro-magnetic wave, there now exists a considerable number of results for such scattering problems, mainly due to Chandler-Wilde and Zhang. In the case of the total field vanishing on the scattering surface, uniqueness of solution is proved in [20] making use of a novel radiation condition first introduced in [11] and further investigated in [19].

To prove existence of solution, the well established boundary integral equation method is employed. However, as opposed to the bounded obstacle case, the operators in the resulting boundary integral equation are no longer compact and thus the Fredholm Alternative cannot be applied to establish surjectivity from injectivity. Thus, Chandler-Wilde, Ross and Zhang [15] had to employ a much more sophisticated solvability theory (see [17] and references contained therein) to prove existence of solution.

Similar uniqueness and existence results are known for acoustic or electromagnetic scattering problems involving an impedance boundary condition on a rough surface [47], inhomogeneous layers [19,48] or rough interfaces [21]. However, no results have been established for the case of an incident elastic wave. This thesis is a contribution towards filling this gap by rigorously establishing uniqueness and existence of solution to the problem of scattering an elastic wave by a rough surface in the case of the total displacement vanishing on the surface, and establishing well posed integral equation formulations for such problems.

1.1 The Problem of Elastic Wave Scattering by a Rough Surface

The propagation of time harmonic waves with circular frequency ω in an elastic solid with Lamé constants μ, λ ($\mu > 0$, $\lambda + \mu \geq 0$) is governed by the *Navier* equation,

$$\mu \, \Delta \, \mathbf{u} + (\lambda + \mu) \, \mathrm{grad} \, \mathrm{div} \, \mathbf{u} + \omega^2 \, \mathbf{u} = 0. \tag{1.1}$$

We will consider elastic waves propagating in an infinite domain $\Omega \subset \mathbb{R}^2$, bounded by a rough surface S, given as the graph of a function $f \in C^{1,\alpha}(\mathbb{R})$. The following scattering problem will be investigated:

> *Scattering Problem:* Given an incident field $\mathbf{u}^{inc}$ that is a solution to (1.1) in Ω, find the scattered field $\mathbf{u}$ such that $\mathbf{u}^{inc} + \mathbf{u} = 0$ on S.

Mathematically, we will formulate this scattering problem as a boundary value problem for a vector field $\mathbf{u} \in [C^2(\Omega) \cap C(\bar{\Omega})]^2$, consisting in the first instance of the Navier equation and the Dirichlet boundary conditions on S. However, this formulation will not be well posed without some further assumptions on the solution. Additionally, a vertical growth condition has to be imposed, and, more importantly, a suitable radiation condition. How this condition is to be formulated is far from clear *a priori* and the question will be investigated in some detail in Chapter 4. This discussion eventually leads to the complete formulation of the boundary value problem for $\mathbf{u}$ as Problem 4.15.

To establish existence of solution to the scattering problem, the boundary integral equation method will be used. We will make an ansatz for the scattered field as a potential of the form

$$\mathbf{u}(\mathbf{x}) = \int_S \mathbf{K}(\mathbf{x}, \mathbf{y}) \, \phi(\mathbf{y}) \, ds(\mathbf{y}), \qquad \mathbf{x} \in \Omega, \tag{1.2}$$

where $\phi \in [BC(S)]^2$, the space of bounded and continuous vector valued functions on S. This ansatz then leads to an integral equation for ϕ, solvability of which has to be proved. However, a number of difficulties arise: The first of these is the suitable choice of the matrix kernel $\mathbf{K}$ in (1.2). For the integral to be well defined for every bounded and continuous vector-valued density ϕ, we have to require that

$$\mathbf{K}(\mathbf{x}, \mathbf{y}) = O(|\mathbf{y}|^{-p}), \qquad \mathbf{y} \in S, \ |\mathbf{y}| \to \infty, \tag{1.3}$$

for every $\mathbf{x} \in \Omega$ and some $p > 1$. Thus the free field Green's tensor, the standard kernel in potential theory, which satisfies (1.3) only for $p \leq 1/2$, is not an appropriate choice.

The second difficulty is that the integral operators in the arising integral equation cannot be expected to be compact operators on the space of bounded and continuous functions. To be able to still deduce existence of solution to the integral equation from uniqueness of solution thus requires a much more sophisticated argument than in the bounded obstacle case.

1.2 Main Results

The discussion starts in Chapter 2 with a presentation of linearized elasticity theory, the foundation of much of what is to follow. Subsequently, the regularity of solutions to the Navier equation (1.1) *up to the boundary* is investigated, making use of regularity results for weak solutions to systems of elliptic partial differential equations which are presented in the appendix.

The last two sections of Chapter 2 are devoted to the topic of matrices of fundamental solutions to the Navier equation. The fundamental solutions presented are the free field Green's tensor Γ and the Green's tensor, $\Gamma_{D,h}$, for a half plane with a rigid boundary. The most important result of this discussion is the proof, in Theorem 2.13, that $\Gamma_{D,h}$ satisfies (1.3) with $p = 3/2$.

The object of Chapter 3 is the investigation of the properties of elastic single- and double-layer potentials on rough surfaces defined using the fundamental solution $\Gamma_{D,h}$. We will start by showing that when using the *pseudo stress operator* to define the kernel of the double-layer potential, this kernel is weakly singular. As the next step we review regularity results for elastic single- and double-layer potentials on a bounded, closed surface defined using the free field Green's tensor, Γ. These results are, in principle, well known. However, the presentation given is novel in the sense that emphasis is laid on the uniformity of the regularity estimates with respect to boundary curves sharing certain elementary geometrical properties. This uniformity property is the key to applying the results for closed boundary curves to prove similar results for potentials defined on rough surfaces, using $\Gamma_{D,h}$ as the matrix kernel. These regularity results, presented as Theorems 3.11 and 3.12, are the main results of this chapter.

Full attention can then finally be paid to the rough surface scattering problem. The first goal here is to find an appropriate radiation condition for such a problem. This condition, termed the *upward propagating radiation condition* (UPRC), is introduced in Definition 4.9 and, subsequently, a thorough investigation of its properties is undertaken. The most important results are given in Theorem 4.12, stating a number of equivalent formulations of the UPRC and establishing how it generalises and can be characterised in terms of other, standard radiation conditions.

The UPRC is then used in the boundary value problem formulation of the rough surface scattering problem as Problem 4.15. Apart from the Navier equation, the Dirichlet boundary conditions and the UPRC, this boundary value problem formulation also includes a growth condition ensuring that the solution remains bounded in all horizontal strips. The remainder of Chapter 4 is then devoted to proving uniqueness of solution to Problem 4.15. After some long and difficult arguments, this goal is finally accomplished in Theorem 4.22, one of the central results of the present thesis.

Chapter 5 is devoted to proving existence of solution to Problem 4.15, thus showing that the problem is well posed. The boundary integral equation method is employed for this purpose: Making a Brakhage/Werner [8] type ansatz for the solution as a combined double- and single-layer potential the problem is reduced to proving solvability of the resulting integral equation (5.2). Equation (5.2) is of the second kind and the matrix kernel has a weak singularity but the range of integration is infinite and so the integral operators are not compact.

The proof is based on a solvability theory for operator equations developed by Chandler-Wilde, Ross and Zhang [10,16,17,22,42]. However, differing significantly from the approach in these papers, solvability is first shown for the adjoint equation, first in the space of bounded and continuous functions and then in subspaces obtained by introducing a weighted norm. From this result, solvability of equation (5.2) is deduced by a duality argument, yielding existence of solution to Problem 4.15 in Theorem 5.24. In the framework of the solvability theory employed, this approach is new. As a valuable consequence, it is shown in the last section of Chapter 5 how the approach can be used to prove solvability of the integral equation not only in the space of bounded and continuous functions but also in all L^p spaces.

1.3 Notes on Notation

Throughout this thesis, all vectors and vector fields will be denoted either in bold type or, in the case of surface densities, by Greek letters. Matrices or matrix functions will be denoted either by bold capital letters or by capital Greek letters.

For any set $\mathcal{S} \subset \mathbb{R}^m$ ($m \in \mathbb{N}$) denote by $BC(\mathcal{S})$ the set of bounded and continuous, complex valued functions on $\mathcal{S}$. The set $BC(\mathcal{S})$ is a Banach space under the supremum norm $\|\cdot\|_{\infty;\mathcal{S}}$.

Let u denote a scalar function defined on a bounded domain $D \subset \mathbb{R}^n$. For $\alpha \in (0,1]$, we introduce the quantity

$$[u]_{\alpha;D} := \sup_{\mathbf{x},\mathbf{y}\in D} \frac{|u(\mathbf{x}) - u(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\alpha},$$

which, in general, may be infinite. The space of Hölder continuous functions is defined as

$$C^{\alpha}(\bar{D}) := \{u \in C(\bar{D}) : [u]_{\alpha;D} < \infty\}.$$

It is a Banach space with the norm

$$\|u\|_{\alpha;D} := \|u\|_{\infty;D} + [u]_{\alpha;D}.$$

We obtain similar spaces of k-times continuously differentiable functions by the definitions

$$C^{k,\alpha}(\bar{D}) := \{u \in C^k(\bar{D}) : [D^k u]_{\alpha;D} < \infty\}$$

and

$$\|u\|_{k,\alpha;D} := \sum_{j=0}^{k} \|D^j u\|_{\infty;D} + [D^k u]_{\alpha;D}, \tag{1.4}$$

where $D^k u$, $k \in \mathbb{N}$, denotes that the maximum of the corresponding norm or semi-norm is to be taken with respect to all k-th partial derivatives of u.

We extend this notion to unbounded domains in the following way: For $\mathcal{S} \subset \mathbb{R}^n$, define

$$\mathcal{V}^{k,\alpha}(\mathcal{S}) := \{u : u \in C^{k,\alpha}(\bar{D}) \text{ for any domain } D \subset\subset \mathcal{S}\},$$

and

$$C^{k,\alpha}(\mathcal{S}) := \{u : u \in \mathcal{V}^{k,\alpha}(\mathcal{S}) \text{ and } \sup_{D \subset\subset \mathcal{S}} \|u\|_{k,\alpha;D} < \infty\}.$$

Here, the notation $D \subset\subset \mathcal{S}$ denotes that the closure of D is a compact subset of $\mathcal{S}$. We also introduce a norm on $C^{k,\alpha}(\mathcal{S})$, by defining, for $u \in C^{k,\alpha}(\mathcal{S})$,

$$\|u\|_{k,\alpha;\mathcal{S}} := \sup_{D \subset\subset \mathcal{S}} \|u\|_{k,\alpha;D},$$

and remark that $C^{k,\alpha}(\mathcal{S})$ is a Banach space with this norm. For an equivalent definition of this norm, (1.4) can be extended to $\mathcal{S}$ for all functions $u \in C^{k,\alpha}(\mathcal{S})$.

We will also make use of the standard Sobolev space $H^1(D)$ for any open set $D \subset \mathbb{R}^n$ and $H^{1/2}(\partial D)$ provided, the boundary of D is smooth enough (see [30, pp. 114] for details). The notations $H^1_{loc}(\mathcal{S})$ and $H^{1/2}_{loc}(\mathcal{S})$ will denote functions that elements of $H^1(D)$ and $H^{1/2}(D)$ for any $D \subset\subset \mathcal{S}$, respectively.

All these definitions generalise to m-vectors $([BC(\mathcal{S})]^m, \dots)$ by requiring all m components to be in the corresponding scalar set. In the vector case, the norms are to be understood as sums of the scalar norms of the components.

Mostly vectors and vector fields in $\mathbb{R}^2$ will be considered. For such vector fields, in addition to the usual differential operators grad$\cdot$ and div$\cdot$, we will make use of

$$\operatorname{grad}^{\perp} u := \left(\frac{\partial u}{\partial x_2}, -\frac{\partial u}{\partial x_1}\right)^{\top} \qquad \text{and} \qquad \operatorname{div}^{\perp} \mathbf{u} := \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}.$$

We remark that these are related to the differential operator curl $\cdot$ in the following way: define $\mathbf{v} := (0,0,u)^\top$ and $\mathbf{w} := (u_1, u_2, 0)^\top$. Then $(\operatorname{grad}^\perp u, 0)^\top = \operatorname{curl} \mathbf{v}$ and $\operatorname{div}^\perp \mathbf{u} = -(\operatorname{curl} \mathbf{w})_3$.

The scattering surface will be represented throughout as the graph of a function $f \in C^{1,\alpha}(\mathbb{R})$. The domain above this surface will be denoted by

$$\Omega := \{\mathbf{x} = (x_1, x_2)^\top \in \mathbb{R}^2 : x_2 > f(x_1)\},$$

and we set $S := \partial\Omega = \{\mathbf{x} \in \mathbb{R}^2 : x_2 = f(x_1)\}$. For $A > 0$, we also introduce

$$S(A) := \{\mathbf{x} \in S : |x_1| < A\}.$$

The normal $\mathbf{n}$ on S will always be assumed to be pointing into Ω.

Throughout the thesis, the letters h and H will frequently be used to denote certain real numbers. As a convention, there will then usually hold $h < \inf f$ and $H > \sup f$.

For $a \in \mathbb{R}$, we also introduce the sets

$$\begin{aligned} U_a &:= \{\mathbf{x} \in \mathbb{R}^2 : x_2 > a\}, \\ T_a &:= \{\mathbf{x} \in \mathbb{R}^2 : x_2 = a\}, \\ D_a &:= \{\mathbf{x} \in \Omega : x_2 < a\}, \\ \gamma(a, A) &:= \{\mathbf{x} \in \Omega : |x_1| = A, x_2 < a\}. \end{aligned}$$

Furthermore, the sets $T_a(A)$ and $D_a(A)$ are defined analogously to $S(A)$. The normals on T_a and $T_a(A)$ are assumed to be pointing into U_a, those on ∂D_a as well as $\partial D_a(A)$ and $\gamma(a, A)$ to be pointing out of D_a and $D_a(A)$ respectively.

Throughout the thesis, use will be made of the Hankel functions of the first kind and order $n \in \mathbb{N}$,

$$H_n^{(1)}(t) := J_n(t) + i\, Y_n(t), \qquad t \in (0, \infty),$$

where J_n and Y_n denote the Bessel functions of first and second kind respectively. The asymptotic decay rate of the Hankel functions and their derivatives as $t \to \infty$ is, for example, as given in [23]: For fixed $n \in \mathbb{N}$,

$$\left.\begin{aligned} H_n^{(1)}(t) &= \sqrt{\frac{2}{\pi t}}\, \mathrm{e}^{i(t - n\pi/2 - \pi/4)} \left\{ 1 + O\left(\frac{1}{t}\right) \right\}, \\ H_n^{(1)\prime}(t) &= \sqrt{\frac{2}{\pi t}}\, \mathrm{e}^{i(t - n\pi/2 + \pi/4)} \left\{ 1 + O\left(\frac{1}{t}\right) \right\}, \end{aligned}\right. \qquad t \to \infty \tag{1.5}$$

From the definitions of the Bessel functions as given in [23], it is clear that the singular behaviour of $H_0^{(1)}(t)$ as $t \to 0$ is

$$H_0^{(1)}(t) = \frac{2i}{\pi} \left(\log \frac{t}{2} + C \right) + 1 + O\left(t^2 \log t\right), \qquad t \to 0, \tag{1.6}$$

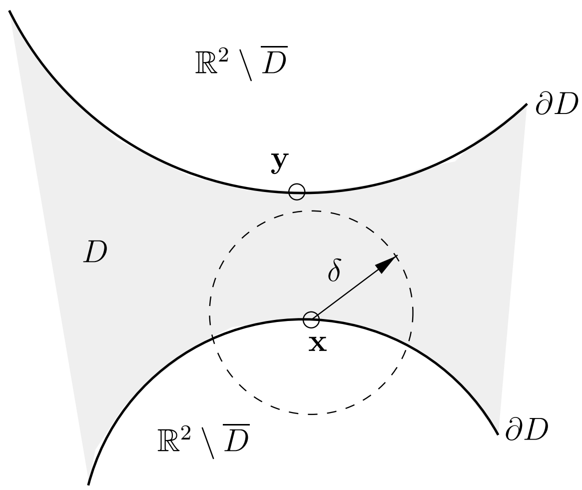

Figure 1.1: Property A_δ for a domain with a boundary closing back in on itself

where C denotes Euler's constant.

We close this section with some remarks on *domains of class* $C^{1,\alpha}$, $\alpha \in (0, 1]$. Adopting the definition given in [23, page 15], a domain $D \subset \mathbb{R}^2$ is said to be of class $C^{1,\alpha}$ if for any point $\mathbf{x} \in \partial D$ there exists neighbourhood $V_\mathbf{x}$ with the following properties: the intersection $V_\mathbf{x} \cap \bar{D}$ can be mapped bijectively onto the half circle $\{\mathbf{y} \in \mathbb{R}^2 : |\mathbf{y}| < 1, y_2 \geq 0\}$; this mapping and its inverse are continuously differentiable and the derivatives are α-Hölder continuous; and the intersection $V_\mathbf{x} \cap \partial D$ is mapped onto the line $\{\mathbf{y} \in \mathbb{R}^2 : |\mathbf{y}| < 1, y_2 = 0\}$. Obviously, the domain Ω is in this class.

If we restrict our attention to a bounded, simply connected domain D, then the surface can be parametrised in terms of the arc length s by a function $\psi : \mathbb{R} \to \mathbb{R}^2$, with $\psi \in [C^{1,\alpha}(\mathbb{R})]^2$, and ψ $|\partial D|$-periodic, where $|\partial D| := \int_{\partial D} ds$ denotes the circumference of D. We define the quantity

$$H_\alpha(D) := \sup_{s,s' \in \mathbb{R}} \frac{|\psi'(s) - \psi'(s')|}{|s - s'|^\alpha}.$$

We note that, in the case $\alpha = 1$, the second distributional derivative of ψ is in $[L^\infty(\mathbb{R})]^2$ and thus the curvature $\kappa(\mathbf{x})$ at $\mathbf{x} \in \partial D$ is defined almost everywhere on ∂D. In this case, $|\kappa(\mathbf{x})| \leq H_1(D)$ holds almost everywhere on ∂D.

For $\mathbf{x}$, $\mathbf{y} \in \partial D$ close enough, let $\partial D(\mathbf{x}, \mathbf{y})$ denote the shorter arc of ∂D connecting $\mathbf{x}$ and $\mathbf{y}$. Then, if D is of class $C^{1,\alpha}$, it will have the following property for some parameter $\delta > 0$:

> **Property $\mathbf{A}_\delta$:** *For all* $\mathbf{x}$, $\mathbf{y} \in \partial D$ *with* $|\mathbf{x}-\mathbf{y}| < \delta$ *there holds* $|\mathbf{x}'-\mathbf{y}| < \delta$ *for all* $\mathbf{x}' \in \partial D(\mathbf{x}, \mathbf{y})$.

In Figure 1.1, this property is illustrated for a domain with a boundary that closes back in on itself. Particularly, the parameter δ has to be chosen such that a circle of radius δ around any point $\mathbf{x} \in \partial D$ contains only one connected part of ∂D.

At certain points in the arguments, we will want to show certain estimates hold with the same constants for all bounded, simply connected domains of class $C^{1,\alpha}$ that share certain geometrical properties. For this purposes, for $\alpha \in (0,1]$, and κ_0, δ, $M > 0$ define

$$\begin{aligned}\mathcal{D}_{\alpha,\kappa_0,\delta,M} \quad := \quad & \{D \subset \mathbb{R}^2 \text{ of class } C^{1,\alpha}, \text{ simply connected and bounded,} \\ & D \text{ satisfies } A_\delta,\ H_\alpha(D) < \kappa_0,\ |\partial D| < M\}.\end{aligned}$$

Similarily, certain results will be shown uniformly with respect to classes of functions f defining the boundary $\partial\Omega$. Thus, for $\alpha \in (0,1]$, $c \in \mathbb{R}$ and $M > 0$, define

$$B_{\alpha,c,M} := \{f \in C^{1,\alpha}(\mathbb{R}) : \|f\|_{1,\alpha;\mathbb{R}} \leq M \text{ and } \inf f \geq c\}.$$

Chapter 2

Time Harmonic Waves in Linearized Elasticity

The basis of the present investigation into scattering of elastic waves by rough surfaces is the theory of linearized elasticity. In this chapter, we will present the fundamental equations of this theory and derive the Navier equation which governs the propagation of time-harmonic waves in an elastic medium. The study of solutions to this equation is then continued by deriving regularity results *up to the boundary* based on regularity estimates for systems of elliptic partial differential equations.

Subsequently we will introduce the matrices of fundamental solutions for the Navier equation in free field conditions and for a half space with a rigid boundary. These fundamental solutions will have a prominent role in all later chapters of this thesis, as they are at the heart of the definition of elastic potentials and also of the new radiation condition to be introduced in Chapter 4.

2.1 Linearized Elasticity Theory

The propagation of waves in an elastic solid with Lamé constants μ, λ ($\mu > 0$, $\lambda + \mu \geq 0$) and density ρ in three-dimensional space is governed by Hooke's law

$$\sigma_{jk} = \lambda \operatorname{div} \mathbf{u}\, \delta_{jk} + \mu \left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right), \qquad j,k = 1,2,3, \tag{2.1}$$

and, in the absence of exterior forces, by the equations of motion

$$\sum_{k=1}^{3} \frac{\partial \sigma_{jk}}{\partial x_k} - \rho \frac{\partial^2 u_j}{\partial t^2} = 0, \qquad j = 1,2,3. \tag{2.2}$$

Here, the vector field $\mathbf{u}$ denotes the displacement and (σ_{jk}) the stress tensor in $\mathbb{R}^3$. We will assume that the density ρ is constant throughout the medium, say $\rho \equiv 1$.

Throughout, only time-harmonic waves with circular frequency $\omega > 0$ will be considered, i. e. all fields are assumed to have a time dependence $\mathrm{e}^{-i\omega t}$. It is common to suppress this time dependence and then, using the same symbols as before, equation (2.2) can be rewritten as

$$\sum_{k=1}^{3} \frac{\partial \sigma_{jk}}{\partial x_k} + \omega^2 u_j = 0, \qquad j = 1, 2, 3. \tag{2.3}$$

Inserting the components of (σ_{jk}) as given by (2.1) into (2.3) then yields the Navier equation

$$\mu\, \Delta \mathbf{u} + (\lambda + \mu)\, \mathrm{grad}\, \mathrm{div}\, \mathbf{u} + \omega^2\, \mathbf{u} = 0. \tag{2.4}$$

Throughout this thesis, we will be considering scattering surfaces which are invariant in one coordinate direction, say in the direction of the x_3-axis, and we will also assume that all waves are propagating perpendicular to that direction, i. e. the fields do not depend on x_3. In these situations, the system (2.4) separates into a two-dimensional system and a scalar equation: Defining $\tilde{\mathbf{u}} := (u_1, u_2)^\top$, we obtain

$$\mu\, \Delta \tilde{\mathbf{u}} + (\lambda + \mu)\, \mathrm{grad}\, \mathrm{div}\, \tilde{\mathbf{u}} + \omega^2\, \tilde{\mathbf{u}} = 0, \tag{2.5}$$

$$\Delta u_3 + \frac{\omega^2}{\mu}\, u_3 = 0. \tag{2.6}$$

Equation (2.6) is, of course, the Helmholtz equation; the analysis of this equation is well understood and we shall not consider it here. For the special case of 2D scattering by rough surfaces we refer the reader to the papers [15, 19, 20] and the references contained therein.

Equation (2.5) is again of the same form as (2.4), only now in two dimensions. It is in the form (2.5) that we will consider the Navier equation from now on, replacing the notation $\tilde{\mathbf{u}}$ by $\mathbf{u}$ again, for convenience: the scattering problem will be treated as a problem of plane strain. For simplicity, we also introduce the notation

$$\Delta^* \mathbf{u} := \mu\, \Delta \mathbf{u} + (\lambda + \mu)\, \mathrm{grad}\, \mathrm{div}\, \mathbf{u}.$$

Because of the assumptions on the Lamé constants ($\mu > 0$ and $\lambda + \mu \geq 0$), it is easy to show that the Navier equation is uniformly strictly elliptic in any domain in $\mathbb{R}^2$. Thus we have the following regularity result, which shall be used extensively without further reference:

Lemma 2.1 *Let Ω denote a domain in $\mathbb{R}^2$ and $\mathbf{u} \in [C^2(\Omega)]^2$ a solution to the Navier equation. Then $\mathbf{u} \in [C^\infty(\Omega)]^2$.*

Proof: The assertion follows from L^2-estimates in [26] and subsequent applications of Sobolev's imbedding theorem. ∎

An important tool in the analysis of the Navier equation are the Lamé potentials. We introduce the wave numbers for compressional and shear waves,

$$k_p := \frac{\omega}{\sqrt{2\mu + \lambda}} \qquad \text{and} \qquad k_s := \frac{\omega}{\sqrt{\mu}}$$

respectively and define the Lamé potentials by

$$\Psi_p := -\frac{1}{k_p^2} \operatorname{div} \mathbf{u} \qquad \text{and} \qquad \Psi_s := -\frac{1}{k_s^2} \operatorname{div}^\perp \mathbf{u}. \tag{2.7}$$

Some important properties of these potentials are listed in the following lemma.

Lemma 2.2 *Let Ω denote a domain in $\mathbb{R}^2$ and $\mathbf{u} \in [C^2(\Omega)]^2$ a solution to the Navier equation. Then,*

$$\begin{aligned} \Delta \Psi_p + k_p^2 \Psi_p &= 0, \\ \Delta \Psi_s + k_s^2 \Psi_s &= 0, \end{aligned}$$

and

$$\mathbf{u} = \operatorname{grad} \Psi_p + \operatorname{grad}^\perp \Psi_s.$$

Proof: We recall

$$\operatorname{div} \Delta \mathbf{u} = \Delta \operatorname{div} \mathbf{u} = \operatorname{div} \operatorname{grad} \operatorname{div} \mathbf{u}.$$

Thus, by applying $\operatorname{div} \cdot$ to (2.4), we obtain

$$(2\mu + \lambda)\, \Delta \operatorname{div} \mathbf{u} + \omega^2 \operatorname{div} \mathbf{u} = 0.$$

The analogous equation for Ψ_s is obtained by applying $\operatorname{div}^\perp \cdot$ and noting that $\operatorname{div}^\perp \operatorname{grad} \operatorname{div} \mathbf{u} = 0$ for any three times continuously differentiable vector field. Finally, we have that

$$\Delta \mathbf{u} - \operatorname{grad} \operatorname{div} \mathbf{u} = \operatorname{grad}^\perp \operatorname{div} \mathbf{u},$$

and thus, by (2.4),

$$\begin{aligned} \mathbf{u} &= -\frac{\mu}{\omega^2} \Delta \mathbf{u} - \frac{\lambda + \mu}{\omega^2} \operatorname{grad} \operatorname{div} \mathbf{u} \\ &= -\frac{1}{k_s^2} \Delta \mathbf{u} - \frac{1}{k_p^2} \operatorname{grad} \operatorname{div} \mathbf{u} + \frac{1}{k_s^2} \operatorname{grad} \operatorname{div} \mathbf{u} \\ &= -\frac{1}{k_p^2} \operatorname{grad} \operatorname{div} \mathbf{u} - \frac{1}{k_s^2} \operatorname{grad}^\perp \operatorname{div}^\perp \mathbf{u}. \end{aligned}$$

This completes the proof. ∎

Remark 2.3 *The vector fields* $\mathbf{u}_p := \operatorname{grad}\Psi_p$ *and* $\mathbf{u}_s := \operatorname{grad}^\perp \Psi_s$ *are called the* compressional *and* shear *parts of* $\mathbf{u}$ *respectively. This terminology is justified by observing that* $\operatorname{div}^\perp \mathbf{u}_p = 0$ *and* $\operatorname{div} \mathbf{u}_s = 0$*, i.e.* $\mathbf{u}_p$ *is irrotational and* $\mathbf{u}_s$ *is divergence free.*

We follow Kupradze [35] in introducing a *generalised stress tensor* (π_{jk}) by

$$\pi_{jk} := \tilde{\lambda} \operatorname{div} \mathbf{u}\, \delta_{jk} + \mu \frac{\partial u_j}{\partial x_k} + \tilde{\mu} \frac{\partial u_k}{\partial x_j},$$

where $\tilde{\mu}$, $\tilde{\lambda}$ are real numbers satisfying $\tilde{\mu} + \tilde{\lambda} = \lambda + \mu$. In the case $\mu = \tilde{\mu}$ and $\lambda = \tilde{\lambda}$, the generalised stress tensor is identical to the standard stress tensor (σ_{jk}).

Given a curve $\Lambda \subset \mathbb{R}^2$ with unit normal $\mathbf{n}$, the *generalised stress vector* on Λ is defined by

$$\mathbf{Pu} := (\pi_{jk})\,\mathbf{n} = (\mu + \tilde{\mu}) \frac{\partial \mathbf{u}}{\partial \mathbf{n}} + \tilde{\lambda}\, \mathbf{n} \operatorname{div} \mathbf{u} - \tilde{\mu}\, \mathbf{n}^\perp \operatorname{div}^\perp \mathbf{u}.$$

This notion has also been used in the papers [25, 38] and its 3D equivalent in [31]. Its significance and its properties for a special choice of $\tilde{\mu}$ and $\tilde{\lambda}$ will be discussed in detail in Chapter 3 when we discuss the properties of elastic single- and double-layer potentials. For now, we only point out that $\mathbf{Pu}$ is equal to the physical stress vector $\mathbf{Tu}$ for the choice $\tilde{\mu} = \mu$ and $\tilde{\lambda} = \lambda$.

Using the generalised stress vector, the generalised Betti formulae result as a consequence of the divergence theorem:

Lemma 2.4 *Let* $B \subseteq \mathbb{R}^2$ *be a domain in which the divergence theorem holds and let* $\mathbf{n}$ *denote the outward drawn normal on* ∂B*. Then, for vector fields* $\mathbf{v} \in [C^1(\bar{B})]^2$ *and* $\mathbf{w} \in [C^2(\bar{B})]^2$*, the first generalised Betti formula holds:*

$$\int_B \mathbf{v} \cdot \Delta^* \mathbf{w}\, d\mathbf{x} = \int_{\partial B} \mathbf{v} \cdot \mathbf{Pw}\, ds - \int_B \mathcal{E}_{\tilde{\mu},\tilde{\lambda}}(\mathbf{v}, \mathbf{w})\, d\mathbf{x}, \tag{2.8}$$

where

$$\begin{aligned} \mathcal{E}_{\tilde{\mu},\tilde{\lambda}}(\mathbf{v}, \mathbf{w}) \;:=\; & (2\mu + \lambda) \left(\frac{\partial v_1}{\partial x_1} \frac{\partial w_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \frac{\partial w_2}{\partial x_2} \right) + \mu \left(\frac{\partial v_1}{\partial x_2} \frac{\partial w_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \frac{\partial w_2}{\partial x_1} \right) \\ & + \tilde{\lambda} \left(\frac{\partial v_1}{\partial x_1} \frac{\partial w_2}{\partial x_2} + \frac{\partial v_2}{\partial x_2} \frac{\partial w_1}{\partial x_1} \right) + \tilde{\mu} \left(\frac{\partial v_1}{\partial x_2} \frac{\partial w_2}{\partial x_1} + \frac{\partial v_2}{\partial x_1} \frac{\partial w_1}{\partial x_2} \right). \end{aligned}$$

For $\mathbf{v} \in [C^2(\bar{B})]^2$ *the second generalised Betti formula holds:*

$$\int_B \mathbf{v} \cdot \Delta^* \mathbf{v}\, \mathbf{dx} = \int_{\partial B} \mathbf{v} \cdot \mathbf{Pv}\, ds - \int_B \mathcal{E}_{\tilde{\mu},\tilde{\lambda}}(\mathbf{v}, \mathbf{v})\, d\mathbf{x}. \tag{2.9}$$

Finally, for $\mathbf{v}$, $\mathbf{w} \in [C^2(\bar{B})]^2$ *the third generalised Betti formula holds:*

$$\int_B (\mathbf{v} \cdot \Delta^*\, \mathbf{w} - \Delta^*\, \mathbf{v} \cdot \mathbf{w})\, d\mathbf{x} = \int_{\partial B} (\mathbf{v} \cdot \mathbf{Pw} - \mathbf{Pv} \cdot \mathbf{w})\, ds. \tag{2.10}$$

Proof: As in Kupradze [35] for the three-dimensional case. ■

2.2 Regularity of Solutions

For many of the subsequent considerations, we will need regularity results for solutions to the Navier equation and their derivatives in domains with smooth boundaries. Appendix A gives proofs of such results for weak solutions to elliptic systems. In this section, we will use these results to obtain regularity estimates up to the boundary for classical solutions to the Navier equation that are also weak solutions.

Let us start with a simple interior estimate, however:

Lemma 2.5 *Given a domain $G \subset \mathbb{R}^2$, let $\mathbf{u} \in [L^\infty(G)]^2$ be a solution to the Navier equation (2.4) in G in a distributional sense. Assume $G' \subset\subset G$ and set $d := d(\partial G', \partial G)$. Then, $\mathbf{u} \in [C^1(\overline{G'})]^2$ and for all $\mathbf{x} \in G'$,*

$$|\mathrm{grad}\, u_k(\mathbf{x})| \leq C\,(1 + d^{-1})\,\|\mathbf{u}\|_{\infty;G} \qquad (k = 1, 2),$$

where C is only dependent on μ, λ and ω.

Proof: Application of estimates in Fichera [26] and Sobolev's Imbedding Theorem. ■

By applications of this result we immediately obtain the following corollary:

Corollary 2.6 *Given a domain $G \subset \mathbb{R}^2$, let $(\mathbf{v}_n) \subset [L^\infty(G)]^2$ be a sequence of solutions to the Navier equation in G and, for some vector field $\mathbf{v}$, suppose that $\mathbf{v}_n(\mathbf{x}) \to \mathbf{v}(\mathbf{x})$ uniformly on compact subsets of G. Then $\mathbf{v} \in [C^2(G)]^2$ and is a solution to the Navier equation in G.*

Now we want to establish regularity up to the boundary, using the results of Appendix A. Given a compact subdomain D of Ω and following Definition A.1, a vector field $\mathbf{u} \in [H^1(D)]^2$ is said to be a *weak solution* to the Navier equation if, for any test field $\mathbf{v} \in [H^1_0(D)]^2$, the equation

$$\int_D \left\{ \mu \sum_{j=1}^{2} \mathrm{grad}\, u_j \cdot \mathrm{grad}\, v_j + (\lambda + \mu)\,\mathrm{div}\,\mathbf{u}\,\mathrm{div}\,\mathbf{v} - \omega^2\,\mathbf{u} \cdot \mathbf{v} \right\} d\mathbf{x} = 0$$

holds (see also [46]).

To apply the regularity results derived in the appendix to classical solutions to the Navier equation in Ω, we now construct a set of bounded sub-domains of class $C^{1,\alpha}$ by the following procedure (illustrated in Figure 2.1). We set , for some $H > \sup f$,

$$\rho := \frac{H - \sup f}{3}.$$

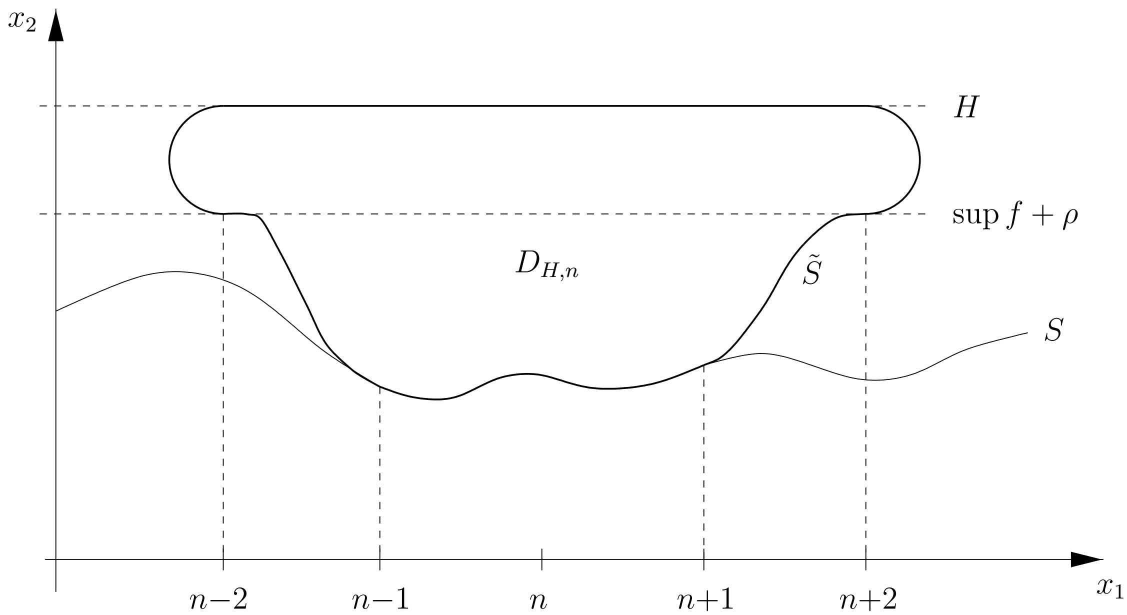

Figure 2.1: Construction of the domain $D_{H,n}$

We also introduce a C^∞ function χ with the following properties: $\chi(s) = 0$ for $s \le \varepsilon$, $\chi(s) = 1$ for $s \ge 1 - \varepsilon$ for some ε, $1/2 > \varepsilon > 0$. We now define the function $\chi_n \in C^{1,\alpha}(\mathbb{R})$ by

$$\chi_n(s) := \begin{cases} \chi(s - n + 2), & s < n - 1, \\ 1, & n - 1 \le s \le n + 1, \\ \chi(n + 2 - s), & n + 1 < s, \end{cases}$$

and finally the function $\tilde{f}$ by

$$\tilde{f}(s) := \chi_n(s)\, f(s) + (1 - \chi_n(s))\, (\sup f + \rho).$$

We now define the domain $D_{H,n}$, $n \in \mathbb{Z}$, as the set of points inside the boundary curve $\partial D_{H,n}$ constructed in the following way:

- between the points $(n-2, \tilde{f}(n-2))^\top$ and $(n+2, \tilde{f}(n+2))^\top$, $\partial D_{H,n}$ is identical to $\tilde{S} := \{\mathbf{x} \in \mathbb{R}^2 : x_2 = \tilde{f}(x_1)\}$,
- outside this section, $\partial D_{H,n}$ is continued as two half circles with radius ρ,
- the two half circles are connected by a straight line.

Thus, $D_{H,n}$ has the portion of S between $n - 1$ and $n + 1$ as part of its boundary, $D_{H,n} \subset D_H$ for all $n \in \mathbb{Z}$, $D_H = \bigcup_{n\in\mathbb{Z}} D_{H,n}$ and there exist numbers κ_0, δ, M such that $D_{H,n} \in \mathcal{D}_{\alpha,\kappa_0,\delta,M}$ for all $n \in \mathbb{Z}$. Moreover, as for $\mathbf{x}$, $\mathbf{y} \in D_H$ with $|\mathbf{x}-\mathbf{y}| \leq 1$ there always exists a number $n_0 \in \mathbb{Z}$ such that $\mathbf{x}$, $\mathbf{y} \in D_{H,n_0}$, it follows that there is a constant C, only depending on α, such that

$$\|\mathbf{u}\|_{1,\alpha;D_H} \leq C \sup_{n\in\mathbb{Z}} \|\mathbf{u}\|_{1,\alpha;D_{H,n}}$$

for any $\mathbf{u} \in C^{1,\alpha}(\overline{D_H})$. Thus we can now apply the regularity results stated in Theorem A.12 to classical solutions of the Navier equation.

Theorem 2.7 *Let $\mathbf{u} \in [C^2(\Omega) \cap C(\bar{\Omega}) \cap H^1_{loc}(\Omega)]^2$ be a solution to the Navier equation in Ω, bounded in D_H for all $H > \sup f$, and $\mathbf{u} = 0$ on S. Then $\mathbf{u} \in \mathcal{V}^{1,\alpha}(\bar{\Omega})$ and, for any $H > \sup f$, $\mathbf{u} \in C^{1,\alpha}(\overline{D_H})$ with*

$$\|\mathbf{u}\|_{1,\alpha;D_H} \leq C \, \|\mathbf{u}\|_{0;D_H}, \tag{2.11}$$

where C is a constant only depending on λ, μ, ω, H, α, κ_0, δ and M.

Proof: Apply Theorem A.12 with $D' = D_{H,n}$ for all $n \in \mathbb{Z}$. ∎

2.3 The Free-Field Green's Tensor

Central to potential theory is the idea of fundamental solutions; in the case of the Navier equation, matrices of fundamental solutions (MFS) fulfill the same role. The k-th column of an MFS, $\mathcal{M}$, to the Navier equation is a solution to the equation

$$\Delta^*_{\mathbf{x}} \mathcal{M}_{\cdot k}(\mathbf{x},\mathbf{y}) + \omega^2 \, \mathcal{M}_{\cdot k}(\mathbf{x},\mathbf{y}) = -\delta(\mathbf{x}-\mathbf{y}) \, \mathbf{e}_k, \qquad k = 1, 2,$$

for $\mathbf{x}$, $\mathbf{y} \in D$ for some domain $D \subset \mathbb{R}^2$, where δ denotes Dirac's delta distribution and $\mathbf{e}_k$ the k-th cartesian unit coordinate vector. The MFS most commonly used in conjunction with the Navier equation is the free field Green's tensor Γ given by

$$\Gamma(\mathbf{x},\mathbf{y}) := \frac{i}{4\mu} H_0^{(1)}(k_s|\mathbf{x}-\mathbf{y}|) \, \mathbf{I} + \frac{i}{4\omega^2} \nabla_x^\top \nabla_x \left(H_0^{(1)}(k_s|\mathbf{x}-\mathbf{y}|) - H_0^{(1)}(k_p|\mathbf{x}-\mathbf{y}|) \right), \tag{2.12}$$

with $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^2$, $\mathbf{x} \neq \mathbf{y}$, where $H_n^{(1)}(\cdot)$ denotes the Hankel function of the first kind and of order n.

As, for example, Kress points out [34], with the help of the Bessel differential equation it is easy to see that the components of this matrix can be written as

$$\Gamma_{jk}(\mathbf{x},\mathbf{y}) = \frac{i}{4\mu} \left\{ \Phi_1(|\mathbf{x}-\mathbf{y}|) \, \delta_{jk} + \Phi_2(|\mathbf{x}-\mathbf{y}|) \, \frac{(x_j - y_j)(x_k - y_k)}{|\mathbf{x}-\mathbf{y}|^2} \right\} \quad (j, k = 1, 2), \tag{2.13}$$

where, introducing the constant $\tau = k_p/k_s$,

$$\begin{aligned}
\Phi_1(t) &:= H_0^{(1)}(k_s t) - \frac{1}{k_s\,t}\left(H_1^{(1)}(k_s t) - \tau\, H_1^{(1)}(k_p t)\right),\\
\Phi_2(t) &:= \frac{2}{k_s\,t}\,H_1^{(1)}(k_s t) - H_0^{(1)}(k_s t) - \frac{2\tau}{k_s\,t}\,H_1^{(1)}(k_p t) + \tau^2\,H_0^{(1)}(k_p t).
\end{aligned}$$

This formulation is very convenient to analyse the singular behaviour of Γ as $|\mathbf{x} - \mathbf{y}| \to 0$ by applying (1.6). This behaviour and some other well known basic properties of Γ are given in the following theorem.

Theorem 2.8 *The MFS Γ is analytic for $\mathbf{x} \neq \mathbf{y}$. It is symmetric, its columns and rows are solutions to the Navier equation with respect to $\mathbf{x}$ in $\mathbb{R}^2 \setminus \{\mathbf{y}\}$ and with respect to $\mathbf{y}$ in $\mathbb{R}^2 \setminus \{\mathbf{x}\}$. Furthermore, it satisfies, for some constant $C > 0$, the estimate*

$$\max_{j,k=1,2} |\Gamma_{jk}(\mathbf{x}, \mathbf{y})| \le C\ (1 + |\log|\mathbf{x} - \mathbf{y}||)\,. \tag{2.14}$$

The Lamé potential representation of the columns of Γ is given by

$$\Gamma_{\cdot k}(\mathbf{x}, \mathbf{y}) = \mathrm{grad}_{\mathbf{x}}\,\Psi_p^{(k)}(\mathbf{x}, \mathbf{y}) + \mathrm{grad}_{\mathbf{x}}^{\perp}\,\Psi_s^{(k)}(\mathbf{x}, \mathbf{y}), \qquad k = 1, 2, \tag{2.15}$$

where

$$\Psi_p^{(1)}(\mathbf{x}, \mathbf{y}) := -\frac{i}{4\omega^2}\frac{\partial}{\partial x_1} H_0^{(1)}(k_p|\mathbf{x} - \mathbf{y}|), \tag{2.16}$$

$$\Psi_p^{(2)}(\mathbf{x}, \mathbf{y}) := -\frac{i}{4\omega^2}\frac{\partial}{\partial x_2} H_0^{(1)}(k_p|\mathbf{x} - \mathbf{y}|), \tag{2.17}$$

$$\Psi_s^{(1)}(\mathbf{x}, \mathbf{y}) := -\frac{i}{4\omega^2}\frac{\partial}{\partial x_2} H_0^{(1)}(k_s|\mathbf{x} - \mathbf{y}|), \tag{2.18}$$

$$\Psi_s^{(2)}(\mathbf{x}, \mathbf{y}) := +\frac{i}{4\omega^2}\frac{\partial}{\partial x_1} H_0^{(1)}(k_s|\mathbf{x} - \mathbf{y}|). \tag{2.19}$$

Using Fourier transforms, we can obtain an alternative representation of Γ which will be useful in the next section. For functions f depending on x_1 and y_1 through the difference $X_1 = x_1 - y_1$, we will temporarily denote by $\mathcal{F}[f]$ or alternatively by $\hat{f}$ the Fourier transform with respect to $X_1 := x_1 - y_1$, i.e.

$$\mathcal{F}[f](t) = \hat{f}(t) = \int_{-\infty}^{\infty} f(X_1)\,\mathrm{e}^{iX_1 t}\,dX_1.$$

It is well known that

$$\mathcal{F}[\,H_0^{(1)}(k|\mathbf{x} - \mathbf{y}|)\,](t) = \frac{2\,\mathrm{e}^{i\sqrt{k^2 - t^2}|x_2 - y_2|}}{\sqrt{k^2 - t^2}}, \tag{2.20}$$

where the branches of the square root function in the complex plane are chosen such that the imaginary part is non-negative. This will be the convention throughout the thesis. We introduce the notations

$$\gamma_p := \sqrt{k_p^2 - t^2} \qquad \text{and} \qquad \gamma_s := \sqrt{k_s^2 - t^2}.$$

Calculating the Fourier transforms of (2.16)–(2.19) using (2.20) and inserting the result in (2.15) then yields

$$\begin{aligned} \mathcal{F}[\,\Gamma(\mathbf{x},\mathbf{y})\,](t) \;&=\; \frac{i}{2\omega^2}\Bigg\{ \begin{pmatrix} t^2/\gamma_p & -t\,\mathrm{sgn}(x_2-y_2) \\ -t\,\mathrm{sgn}(x_2-y_2) & \gamma_p \end{pmatrix} \mathrm{e}^{i\gamma_p|x_2-y_2|} \\ &\quad + \begin{pmatrix} \gamma_s & t\,\mathrm{sgn}(x_2-y_2) \\ t\,\mathrm{sgn}(x_2-y_2) & t^2/\gamma_s \end{pmatrix} \mathrm{e}^{i\gamma_s|x_2-y_2|} \Bigg\}. \end{aligned} \tag{2.21}$$

By applying the generalised stress operator $\mathbf{P}$ to Γ, we obtain matrix functions $\Pi^{(1)}$ and $\Pi^{(2)}$:

$$\begin{aligned} \Pi^{(1)}_{jk}(\mathbf{x},\mathbf{y}) &:= \left(\mathbf{P}^{(\mathbf{x})}(\Gamma_{\cdot k}(\mathbf{x},\mathbf{y}))\right)_j, \\ \Pi^{(2)}_{jk}(\mathbf{x},\mathbf{y}) &:= \left(\mathbf{P}^{(\mathbf{y})}(\Gamma_{j\cdot}(\mathbf{x},\mathbf{y}))^\top\right)_k. \end{aligned}$$

The properties of these matrix functions, very similar to those of Γ, are listed in the following theorem.

Theorem 2.9

(a) For $\mathbf{y} \in \mathbb{R}^2$, the columns of $\Pi^{(2)}(\cdot,\mathbf{y})$ are solutions to the Navier equation (2.4) in $\mathbb{R}^2 \setminus \{\mathbf{y}\}$.

(b) For $\mathbf{x} \in \mathbb{R}^2$, the rows of $\Pi^{(1)}(\mathbf{x},\cdot)$ are solutions to the Navier equation (2.4) in $\mathbb{R}^2 \setminus \{\mathbf{x}\}$.

(c) For $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^2$, $\mathbf{x} \neq \mathbf{y}$, there holds

$$\Pi^{(2)}(\mathbf{x},\mathbf{y}) = \Pi^{(1)}(\mathbf{y},\mathbf{x})^\top.$$

(d) Let $B \subseteq \mathbb{R}^2$ be a bounded domain in which the divergence theorem holds. Then any solution $\mathbf{u} \in C^2(\bar{B})$ to the Navier equation can be represented as

$$\mathbf{u}(\mathbf{x}) = \int_{\partial B} \Gamma(\mathbf{x},\mathbf{y})\,\mathbf{P}\mathbf{u}(\mathbf{y}) - \Pi^{(2)}(\mathbf{x},\mathbf{y})\,\mathbf{u}(\mathbf{y})\,ds(\mathbf{y})$$

for all $\mathbf{x} \in B$.

Proof: Part (c) follows directly from the definitions of $\Pi^{(j)}$ ($j = 1, 2$) and $\mathbf{P}$ together with Theorem 2.8. The same argument yields that the columns of $\Pi^{(2)}(\cdot, \mathbf{y})$ are solutions to the Navier equation in $\mathbb{R}^2 \setminus \{\mathbf{y}\}$.

Part (b) is now a direct consequence of parts (a) and (c).

Part (d) is finally seen by applying the 3rd generalised Betti formula (2.10) and a standard potential theoretic argument, using the fact that $\Gamma(\mathbf{x}, \mathbf{y})$ has a logarithmic singularity for $|\mathbf{x} - \mathbf{y}| \to 0$. ∎

2.4 The Green's Tensor for the First Boundary Value Problem in a Half Space

As was pointed out in the introduction, and can be proven rigorously from (2.13), the free field Green's tensor Γ satisfies (1.3) only for $p = 1/2$. This asymptotic decay rate as $|\mathbf{x} - \mathbf{y}| \to \infty$ is not sufficient to ensure that integrals of the type

$$\int_S \Gamma(\mathbf{x}, \mathbf{y})\, \phi(\mathbf{y})\, ds(\mathbf{y})$$

exist for all $\phi \in [BC(S)]^2$. Therefore, we will derive and analyse the Green's tensor $\Gamma_{D,h}$ for the first boundary value problem of elasticity in a half space U_h ($h \in \mathbb{R}$). This Green's tensor was first introduced in [4]; this section gives a more detailed presentation of the results of that paper.

Motivated by the form of the corresponding Green's function for acoustical wave propagation, we make an ansatz of the form

$$\Gamma_{D,h}(\mathbf{x}, \mathbf{y}) = \Gamma(\mathbf{x}, \mathbf{y}) - \Gamma(\mathbf{x}, \mathbf{y}'_h) + \mathbf{U}(\mathbf{x}, \mathbf{y}), \qquad \mathbf{x}, \mathbf{y} \in U_h, \mathbf{x} \neq \mathbf{y}, \tag{2.22}$$

with a yet unknown matrix function $\mathbf{U}$. In fact, we will assume that $\mathbf{U}$ only depends on $\mathbf{x}$ and $\mathbf{y}$ through the variables $X_1 = x_1 - y_1$, x_2 and y_2. For fixed $\mathbf{y} \in U_h$, the columns of $\mathbf{U}(\cdot, \mathbf{y})$ have to satisfy

$$\begin{aligned} \Delta^*_{\mathbf{x}} \mathbf{U}_{\cdot,k}(\mathbf{x}, \mathbf{y}) + \omega^2 \mathbf{U}_{\cdot,k}(\mathbf{x}, \mathbf{y}) &= 0, && \mathbf{x} \in U_h \setminus \{\mathbf{y}\}, \\ \mathbf{U}_{\cdot,k}(\mathbf{x}, \mathbf{y}) &= -\Gamma_{\cdot k}(\mathbf{x}, \mathbf{y}) + \Gamma_{\cdot k}(\mathbf{x}, \mathbf{y}'_h), && \mathbf{x} \in \partial U_h, \end{aligned}$$

and they also have to be bounded and represent a wave field propagating away from T_h (this notion will be made mathematically precise in Chapter 4 when we discuss radiation conditions).

We represent $\mathbf{U}$ by its Lamé potentials,

$$\mathbf{U}_{\cdot k}(\mathbf{x}, \mathbf{y}) = \operatorname{grad}_{\mathbf{x}} \Psi^{(k)}_{\mathbf{U},p}(\mathbf{x}, \mathbf{y}) + \operatorname{grad}^{\perp}_{\mathbf{x}} \Psi^{(k)}_{\mathbf{U},s}(\mathbf{x}, \mathbf{y}), \qquad k = 1, 2, \tag{2.23}$$

and conclude by Lemma 2.2 that

$$\Delta_{\mathbf{x}}\Psi^{(k)}_{\mathbf{U},p}(\mathbf{x},\mathbf{y}) + k_p^2\Psi^{(k)}_{\mathbf{U},p}(\mathbf{x},\mathbf{y}) = 0 \qquad \text{and} \qquad \Delta_{\mathbf{x}}\Psi^{(k)}_{\mathbf{U},s}(\mathbf{x},\mathbf{y}) + k_s^2\Psi^{(k)}_{\mathbf{U},p}(\mathbf{x},\mathbf{y}) = 0.$$

Taking the Fourier transform with respect to X_1, we obtain the two ordinary differential equations

$$\frac{d^2}{dx_2^2}\hat{\Psi}^{(k)}_{\mathbf{U},p} + \gamma_p^2\,\hat{\Psi}^{(k)}_{\mathbf{U},p} = 0 \tag{2.24}$$

$$\text{and} \qquad \frac{d^2}{dx_2^2}\hat{\Psi}^{(k)}_{\mathbf{U},s} + \gamma_s^2\,\hat{\Psi}^{(k)}_{\mathbf{U},s} = 0 \tag{2.25}$$

As it is assumed that $\mathbf{U}$ be bounded and represent an outgoing wave field, we select solutions to these equations of the form

$$\hat{\Psi}^{(k)}_{\mathbf{U},p} = A_p^{(k)}(t,y_2)\,\mathrm{e}^{i\gamma_p(x_2-h)} \qquad \text{and} \qquad \hat{\Psi}^{(k)}_{\mathbf{U},s} = A_s^{(k)}(t,y_2)\,\mathrm{e}^{i\gamma_s(x_2-h)}. \tag{2.26}$$

The coefficient functions $A_p^{(k)}$ and $A_s^{(k)}$ can now be calculated from the boundary conditions. Using (2.21), we obtain, for $x_2 = h$,

$$\begin{aligned}
-it\hat{\Psi}^{(1)}_{\mathbf{U},p} + \frac{d}{dx_2}\hat{\Psi}^{(1)}_{\mathbf{U},s} &= 0,\\
\frac{d}{dx_2}\hat{\Psi}^{(1)}_{\mathbf{U},p} + it\hat{\Psi}^{(1)}_{\mathbf{U},s} &= \frac{it}{\omega^2}\left(\mathrm{e}^{i\gamma_p(y_2-h)} - \mathrm{e}^{i\gamma_s(y_2-h)}\right),\\
-it\hat{\Psi}^{(2)}_{\mathbf{U},p} + \frac{d}{dx_2}\hat{\Psi}^{(2)}_{\mathbf{U},s} &= \frac{it}{\omega^2}\left(\mathrm{e}^{i\gamma_p(y_2-h)} - \mathrm{e}^{i\gamma_s(y_2-h)}\right),\\
\frac{d}{dx_2}\hat{\Psi}^{(2)}_{\mathbf{U},p} + it\hat{\Psi}^{(2)}_{\mathbf{U},s} &= 0.
\end{aligned}$$

Inserting the solutions (2.26) yields, after some elementary calculations,

$$\begin{pmatrix} A_p^{(1)} & A_s^{(1)} \\ A_p^{(2)} & A_s^{(2)} \end{pmatrix}(t,y_2) = -\frac{1}{\omega^2\left(\gamma_p\gamma_s + t^2\right)}\begin{pmatrix} t\gamma_s & t^2 \\ -t^2 & t\gamma_p \end{pmatrix}\left(\mathrm{e}^{i\gamma_p(y_2-h)} - \mathrm{e}^{i\gamma_s(y_2-h)}\right).$$

Now, taking inverse Fourier transforms and using (2.23), we finally arrive at

$$\mathbf{U}(\mathbf{x},\mathbf{y}) = -\frac{i}{2\pi\omega^2}\int_{-\infty}^{\infty}\left(M_p(t,\gamma_p,\gamma_s;x_2,y_2) + M_s(t,\gamma_p,\gamma_s;x_2,y_2)\right)\mathrm{e}^{-iX_1t}\,dt, \tag{2.27}$$

with

$$\begin{aligned}
M_p(t,\gamma_p,\gamma_s;x_2,y_2) &:= \frac{\mathrm{e}^{i\gamma_p(x_2+y_2-2h)} - \mathrm{e}^{i(\gamma_p(x_2-h)+\gamma_s(y_2-h))}}{\gamma_p\gamma_s + t^2}\begin{pmatrix} -t^2\gamma_s & t^3 \\ t\gamma_p\gamma_s & -t^2\gamma_p \end{pmatrix},\\
M_s(t,\gamma_p,\gamma_s;x_2,y_2) &:= \frac{\mathrm{e}^{i\gamma_s(x_2+y_2-2h)} - \mathrm{e}^{i(\gamma_s(x_2-h)+\gamma_p(y_2-h))}}{\gamma_p\gamma_s + t^2}\begin{pmatrix} -t^2\gamma_s & -t\gamma_p\gamma_s \\ -t^3 & -t^2\gamma_p \end{pmatrix}.
\end{aligned}$$

From the construction of $\mathbf{U}$, we immediately have the following theorem, listing some basic properties of $\Gamma_{D,h}$:

Theorem 2.10 *For $\mathbf{y} \in U_h$, $\Gamma_{D,h}(\cdot, \mathbf{y}) - \Gamma(\cdot, \mathbf{y}) \in [C^\infty(U_h) \cap C^1(\overline{U_h})]^{2\times 2}$, and the columns of $\Gamma_{D,h}(\cdot, \mathbf{y})$ are solutions to the Navier equation (2.4) in $U_h \setminus \{\mathbf{y}\}$. Furthermore, $\Gamma_{D,h}(\mathbf{x}, \mathbf{y}) = 0$ for $\mathbf{x} \in \partial U_h$.*

We will now address the main advantage of using $\Gamma_{D,h}$ over Γ in rough surface scattering applications, its faster asymptotic decay rate as $|x_1| \to \infty$ in horizontal layers above ∂U_h. For the first two terms in its representation, this is shown in the following lemma:

Lemma 2.11 *For $\mathbf{x}$, $\mathbf{y} \in U_h$, $|\mathbf{x} - \mathbf{y}| \geq 1$ the estimate*

$$\max_{j,k=1,2} |\Gamma_{jk}(\mathbf{x}, \mathbf{y}) - \Gamma_{jk}(\mathbf{x}, \mathbf{y}'_h)| \leq \frac{\mathcal{H}(x_2 - h, y_2 - h)}{|\mathbf{x} - \mathbf{y}|^{3/2}},$$

holds, where $\mathcal{H} \in C(\mathbb{R}^2)$.

Proof: Using (2.13) and the notations $r = |\mathbf{x} - \mathbf{y}|$ and $r' = |\mathbf{x} - \mathbf{y}'_h|$, there holds

$$\Gamma(\mathbf{x}, \mathbf{y}) - \Gamma(\mathbf{x}, \mathbf{y}'_h) = \frac{i}{4\mu} \Bigg\{ (\Phi_1(r) - \Phi_1(r'))\, I$$

$$+ \begin{pmatrix} 0 & 2(2h - y_2)(x_1 - y_1) \\ 2(2h - y_2)(x_1 - y_1) & 0 \end{pmatrix} \frac{\Phi_2(r)}{r^2}$$

$$+ \begin{pmatrix} (x_1 - y_1)^2 & (x_1 - y_1)(x_2 + y_2 - 2h) \\ (x_1 - y_1)(x_2 + y_2 - 2h) & (x_2 + y_2 - 2h)^2 \end{pmatrix} \left(\frac{\Phi_2(r)}{r^2} - \frac{\Phi_2(r')}{r'^2} \right) \Bigg\}.$$

So it obviously suffices to show the estimate for the functions

$$\Phi_1(r) - \Phi_1(r'), \qquad \frac{(x_1 - y_1)\Phi_2(r)}{r^2}, \quad \text{and} \quad (x_1 - y_1)^2 \left(\frac{\Phi_2(r)}{r^2} - \frac{\Phi_2(r')}{r'^2} \right).$$

Using the mean value theorem yields

$$|\Phi_1(r) - \Phi_1(r')| \leq |r - r'| \max_{r \leq t \leq r'} |\Phi'_1(t)| = \frac{4(x_2 - h)(y_2 - h)}{r + r'} \max_{r \leq t \leq r'} |\Phi'_1(t)|$$

and thus the asymptotic decay rate of Hankel functions and their derivatives (1.5) yields the asserted estimate in the first case because of the assumption $|\mathbf{x} - \mathbf{y}| \geq 1$. In the second case, $\frac{x_1 - y_1}{r}$ is bounded and $\frac{\Phi_2(r)}{r}$ has the required decay rate. For the last function, we rewrite

$$\frac{\Phi_2(r)}{r^2} - \frac{\Phi_2(r')}{r'^2} = \frac{\left(\Phi_2(r) - \Phi_2(r') + \frac{4(x_2 - h)(y_2 - h)}{r^2} \Phi_2(r) \right)}{r^2 + 4(x_2 - h)(y_2 - h)}.$$

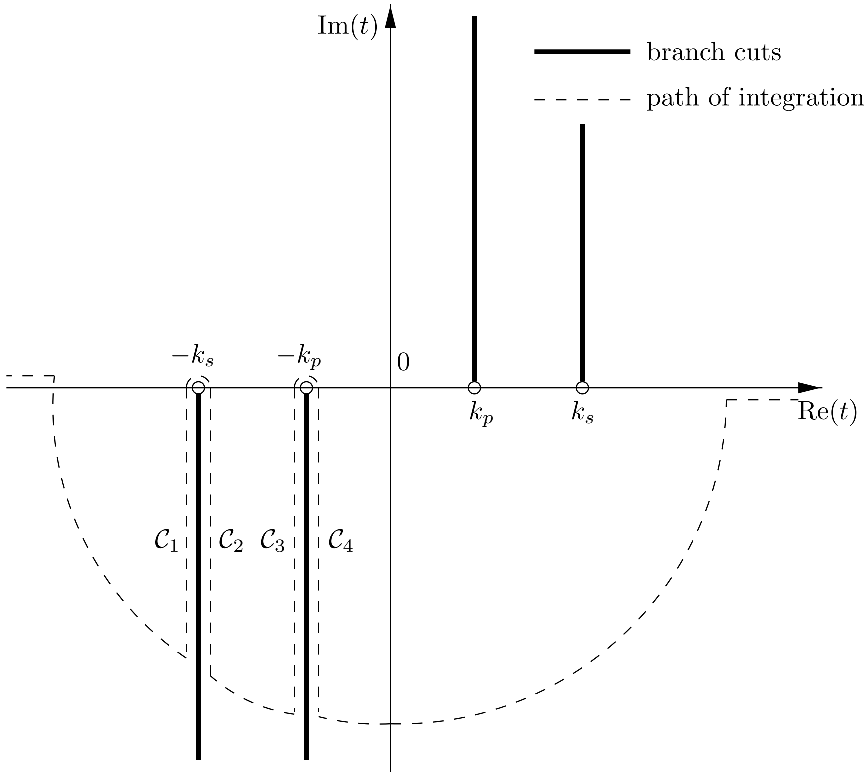

Figure 2.2: The path of integration

Now, $\frac{(x_1-y_1)^2}{r^2+4(x_2-h)(y_2-h)}$ is bounded, $\Phi_2(r)-\Phi_2(r')$ can be estimated in the same way as $\Phi_1(r)-\Phi_1(r')$ above and $\frac{\Phi_2(r)}{r^2}$ decays even faster than required. ■

To prove a similar estimate for $\mathbf{U}$, a more lengthy analysis is required. To obtain alternative representations of the integrals used in (2.27), we deform the path of integration in the complex plane. To this end, branch cuts from $\pm k_p$ and $\pm k_s$, respectively to $\pm k_p \pm i\infty$ and $\pm k_s \pm i\infty$, are introduced. Recall that the branches of the analytic extensions of γ_p and γ_s were chosen such that their imaginary part is non-negative. Note also that the integrands in (2.27) do not have any singularities on the chosen branches of γ_p and γ_s. Restricting ourselves to the case $x_1 > y_1$ for the moment, we deform the path of integration into the lower half plane as illustrated in Figure 2.2.

It is easily seen that the integrals over the arcs vanish as their radius tends to infinity, so only the branch line integrals remain. Denoting the paths of integration along the branch cuts by $\mathcal{C}_1 \cup \mathcal{C}_2$ and $\mathcal{C}_3 \cup \mathcal{C}_4$, as indicated in Figure 2.2, we rewrite $\mathbf{U}$ as

$$\mathbf{U}(\mathbf{x},\mathbf{y}) = I_1 + I_2 + I_3 + I_4, \tag{2.28}$$

with

$$\begin{aligned}
I_1 &:= -\frac{i}{2\pi\omega^2}\int_{\mathcal{C}_1\cup\mathcal{C}_2} M_p(t,\gamma_p,\gamma_s;x_2-h,y_2-h)\,\mathrm{e}^{-iX_1t}\,dt,\\
I_2 &:= -\frac{i}{2\pi\omega^2}\int_{\mathcal{C}_1\cup\mathcal{C}_2} M_s(t,\gamma_p,\gamma_s;x_2-h,y_2-h)\,\mathrm{e}^{-iX_1t}\,dt,\\
I_3 &:= -\frac{i}{2\pi\omega^2}\int_{\mathcal{C}_3\cup\mathcal{C}_4} M_p(t,\gamma_p,\gamma_s;x_2-h,y_2-h)\,\mathrm{e}^{-iX_1t}\,dt,\\
I_4 &:= -\frac{i}{2\pi\omega^2}\int_{\mathcal{C}_3\cup\mathcal{C}_4} M_s(t,\gamma_p,\gamma_s;x_2-h,y_2-h)\,\mathrm{e}^{-iX_1t}\,dt.
\end{aligned}$$

More explicitly, there holds

$$\begin{aligned}
I_1 = -\frac{i}{2\pi\omega^2}\,\mathrm{e}^{iX_1k_s}\int_0^\infty &\{M_p(-k_s-is,\gamma_p|_{\mathcal{C}_2},\gamma_s|_{\mathcal{C}_2};x_2-h,y_2-h)\\
&- M_p(-k_s-is,\gamma_p|_{\mathcal{C}_1},\gamma_s|_{\mathcal{C}_1};x_2-h,y_2-h)\}\mathrm{e}^{-X_1s}\,ds,
\end{aligned}$$

and similar formulae for the other three integrals. Note that $\gamma_s|_{\mathcal{C}_2} = -\overline{\gamma_s|_{\mathcal{C}_1}}$, and $\gamma_p|_{\mathcal{C}_1} = \gamma_p|_{\mathcal{C}_2}$. Using the mean-value theorem, we thus conclude

$$\begin{aligned}
&M_p(-k_s-is,\gamma_p|_{\mathcal{C}_2},\gamma_s|_{\mathcal{C}_2};x_2,y_2) - M_p(-k_s-is,\gamma_p|_{\mathcal{C}_1},\gamma_s|_{\mathcal{C}_1};x_2,y_2)\\
&\qquad = 2\,\mathrm{Re}(\gamma_s|_{\mathcal{C}_1})\,\frac{\partial M_p}{\partial\gamma_s}(-k_s-is,\gamma_p|_{\mathcal{C}_1},\xi;x_2,y_2)
\end{aligned}$$

for some ξ on the line between $\gamma_s|_{\mathcal{C}_1}$ and $\gamma_s|_{\mathcal{C}_2}$. Now, $\frac{\partial M_p}{\partial\gamma_s}(-k_s-is,\gamma_p|_{\mathcal{C}_1},\xi;x_2,y_2)$ is seen to be continuously dependent on s in $[0,\infty)$ and, for some constant C continuously dependent on x_2 and y_2, there holds

$$\left|s^{-1/2}\,\mathrm{Re}(\gamma_s|_{\mathcal{C}_2})\frac{\partial M}{\partial\gamma_s}(-k_s-is,\gamma_p|_{\mathcal{C}_1},\xi;x_2,y_2)\right| \le C$$

for $s\in[0,1]$. Therefore, we can estimate the asymptotic decay rate of I_1 by employing the following lemma with $r=\frac{1}{2}$.

Lemma 2.12 *Assume $q\in C([0,\infty))$ such that $C_1 := \int_0^\infty |q(s)|\mathrm{e}^{-s}\,ds$ exists. For $X>1$, set*

$$I(X) := \int_0^\infty q(s)\,\mathrm{e}^{-Xs}\,ds.$$

Further assume that for some $r>-1$ there exists $C_2>0$ with $|s^{-r}q(s)|\le C_2$ for all $s\in[0,1]$. Then, for $X\ge 1+(r+1)\log X$,

$$|I(X)| \le (C_1+\Gamma(r+1)\,C_2)\,\frac{1}{X^{r+1}}.$$

Proof: We can estimate

$$\left|\int_0^1 q(s)\,\mathrm{e}^{-Xs}\,ds\right| \leq C_2\int_0^1 s^r\mathrm{e}^{-Xs}\,ds \leq C_2\frac{\Gamma(r+1)}{X^{r+1}}.$$

On the other hand, we have

$$\left|\int_1^\infty q(s)\,\mathrm{e}^{-Xs}\,ds\right| \leq \mathrm{e}^{-(X-1)}\int_1^\infty |q(s)|\mathrm{e}^{-s}\,ds \leq \frac{C_1}{X^{r+1}}$$

for all $X \geq 1+(r+1)\log X$. Adding these two estimates yields the assertion. ■

An identical analysis yields this decay rate for the other three integrals in (2.28) and also for the case $x_1 < y_1$. In the latter case the path of integration has to be deformed into the upper half plane. Thus, also recalling Lemma 2.11, the following theorem is proved:

Theorem 2.13 *For $\mathbf{x},\mathbf{y} \in U_h$, $\varepsilon > 0$ and $|x_1 - y_1| \geq \varepsilon$, the estimate*

$$\max_{j,k=1,2} |\Gamma_{D,h,jk}(\mathbf{x},\mathbf{y})| \leq \frac{\mathcal{H}(x_2-h, y_2-h)}{|x_1-y_1|^{3/2}}$$

holds, where $\mathcal{H} \in C(\mathbb{R}^2)$.

Remark 2.14 *We also note that from (2.14) together with Theorem 2.13, we see that there exists some constant $C > 0$ such that*

$$\max_{j,k=1,2} |\Gamma_{D,h,jk}(\mathbf{x},\mathbf{y})| \leq C\,|1+\log|\mathbf{x}-\mathbf{y}|\,| \tag{2.29}$$

for $\mathbf{x}$, $\mathbf{y} \in U_h$. As a consequence, together with an application of Theorem 2.13, we have, for $h < \inf f$ and $h' > \sup f$, that

$$\sup_{\mathbf{x}\in D_{h'}} \int_S \max_{j,k=1,2} |\Gamma_{D,h,jk}(\mathbf{x},\mathbf{y})|^2\,ds(\mathbf{y}) < \infty.$$

For much of the subsequent arguments, the following lemma will be useful:

Lemma 2.15 *For $\mathbf{x}$, $\mathbf{y} \in U_h$, $\mathbf{x} \neq \mathbf{y}$, the following reciprocity relation holds:*

$$\Gamma_{D,h}(\mathbf{x},\mathbf{y}) = \Gamma_{D,h}(\mathbf{y},\mathbf{x})^\top.$$

Proof: From (2.22) and (2.27) by tedious but elementary calculations. ■

Similarly to the case of the free-field's Green's tensor, we can now apply the generalised stress operator to $\Gamma_{D,h}$, thus introducing the matrix functions $\Pi^{(1)}_{D,h}$ and $\Pi^{(2)}_{D,h}$:

$$\begin{aligned}\Pi^{(1)}_{D,h,jk}(\mathbf{x},\mathbf{y}) &:= \left(\mathbf{P}^{(\mathbf{x})}(\Gamma_{D,h,\cdot k}(\mathbf{x},\mathbf{y}))\right)_j\,,\\ \Pi^{(2)}_{D,h,jk}(\mathbf{x},\mathbf{y}) &:= \left(\mathbf{P}^{(\mathbf{y})}(\Gamma_{D,h,j\cdot}(\mathbf{x},\mathbf{y}))^\top\right)_k\,.\end{aligned}$$

Theorem 2.16 *Assume* $\mathbf{x}$, $\mathbf{y} \in U_h$, $\mathbf{x} \neq \mathbf{y}$. *Then,*

(a) Theorem 2.13 holds with $\Gamma_{D,h}$ *replaced by* $\Pi^{(1)}_{D,h}$ *and* $\Pi^{(2)}_{D,h}$ *respectively,*

(b) $\Pi^{(1)}_{D,h}(\mathbf{x},\mathbf{y}) = \Pi^{(2)}_{D,h}(\mathbf{y},\mathbf{x})^\top$,

(c) the columns of $\Pi^{(2)}_{D,h}(\cdot,\mathbf{y})$ *are solutions to the Navier equation in* $U_h \setminus \{\mathbf{y}\}$,

(d) the rows of $\Pi^{(1)}_{D,h}(\mathbf{x},\cdot)$ *are solutions to the Navier equation in* $U_h \setminus \{\mathbf{x}\}$,

(e) Let $B \subset U_h$ *be a bounded domain in which the divergence theorem holds. Then any solution* $\mathbf{u} \in [C^2(\bar{B})]^2$ *to the Navier equation can be represented as*

$$\mathbf{u}(\mathbf{x}) = \int_{\partial B} \Gamma_D(\mathbf{x},\mathbf{y})\,\mathbf{P}\mathbf{u}(\mathbf{y}) - \Pi^{(2)}_D(\mathbf{x},\mathbf{y})\,\mathbf{u}(\mathbf{y})\,ds(\mathbf{y})$$

for all $\mathbf{x} \in B$.

Proof: Part (a) follows from Lemma 2.5. Part (b) is an immediate consequence of Lemma 2.15. Part (c) follows from the definition of the generalised stress vector. Part (d) follows from parts (b) and (c). Finally, (e) holds because of the corresponding relation for Γ (Theorem 2.9) together with Theorem 2.10 and the third generalised Betti formula (2.10). ■

Remark 2.17 *Similarly to (2.29), we also prove that* $\Pi^{(2)}_{D,h}(\mathbf{x},\mathbf{y})$ *remains bounded for* $|\mathbf{x}-\mathbf{y}| \geq \varepsilon > 0$. *Thus, and using Theorem 2.13 and Lemma 2.5, we see for* $H' > H > h$ *and any derivative with respect to* $\mathbf{x}$, $\mathcal{G}$, *of* $\Pi^{(2)}_{D,h}$ *that*

$$\sup_{\mathbf{x}\in U_H\setminus U_{H'}} \int_{T_h} \max_{j,k=1,2} |\mathcal{G}_{jk}(\mathbf{x},\mathbf{y})|\,ds(\mathbf{y}) < \infty.$$

Chapter 3

Elastic Potentials on Rough Surfaces

It is the object of this chapter to establish regularity results for elastic potentials defined on the surface S and mapping properties of related integral operators. Throughout this chapter, we will limit ourselves to surfaces given as the graph of functions $f \in C^{1,1}(\mathbb{R})$. To be able to establish our final results, we will point out a special choice of the values $\tilde{\mu}$ and $\tilde{\lambda}$ in the definition of the generalised stress vector, for which the kernel in the definition of the double-layer potential only has a weak singularity for $\mathbf{x} \to \mathbf{y}$ on S. We will then restate some well known properties of the elastic potentials in the case of a closed boundary curve. These results will differ from the usual formulation, however, in that emphasis will be laid on uniformity with respect to a certain class of boundary curves. Returning to the case of the unbounded surface S, the results for the closed boundary curve will be applied by decomposing the potentials into a smooth part defined on all of S and a singular part with support only on a compact subset of S. Thus we derive similar regularity results for the rough surface potentials.

3.1 Basic Properties of Elastic Potentials

We will start this chapter by introducing the elastic potentials of interest to us. For a vector valued density $\phi \in [BC(S)]^2$, we define an elastic single-layer potential on the rough surface S by

$$\mathbf{v}(\mathbf{x}) := \int_S \Gamma_{D,h}(\mathbf{x}, \mathbf{y})\, \phi(\mathbf{y})\, ds(\mathbf{y}) \qquad \text{for } \mathbf{x} \in U_h \setminus S, \tag{3.1}$$

and a double-layer potential on S by

$$\mathbf{w}(\mathbf{x}) := \int_S \Pi^{(2)}_{D,h}(\mathbf{x},\mathbf{y})\,\phi(\mathbf{y})\,ds(\mathbf{y}) \qquad \text{for } \mathbf{x} \in U_h \setminus S. \tag{3.2}$$

If we assume D to be a bounded, simply connected domain of class $C^{1,1}$, we similarly can, for $\phi \in [C(\partial D)]^2$, define a single-layer potential on ∂D by

$$\mathbf{v}_D(\mathbf{x}) := \int_{\partial D} \Gamma(\mathbf{x},\mathbf{y})\,\phi(\mathbf{y})\,ds(\mathbf{y}), \qquad \mathbf{x} \in \mathbb{R}^2 \setminus \partial D, \tag{3.3}$$

and the elastic double-layer potential

$$\mathbf{w}_D(\mathbf{x}) := \int_{\partial D} \Pi^{(2)}(\mathbf{x},\mathbf{y})\,\phi(\mathbf{y})\,ds(\mathbf{y}), \qquad \mathbf{x} \in \mathbb{R}^2 \setminus \partial D. \tag{3.4}$$

From the properties of the fundamental solutions Γ and $\Gamma_{D,h}$ as well as their derivatives $\Pi^{(2)}$ and $\Pi^{(2)}_{D,h}$, it is clear that all the integrals exist as improper integrals. Furthermore, the following theorem is a standard result:

Theorem 3.1 *The potentials $\mathbf{v}$ and $\mathbf{w}$ are solutions to the Navier equation in $U_h \setminus \bar{\Omega}$ and in Ω. The potentials $\mathbf{v}_D$ and $\mathbf{w}_D$ are solutions to the Navier equation in D and in $\mathbb{R}^2 \setminus \bar{D}$.*

3.2 The Pseudo Stress Operator

In Section 2.1 the generalised stress operator $\mathbf{P}$ was introduced. We will now investigate what influence the parameters $\tilde{\mu}$ and $\tilde{\lambda}$ in its definition have on the singular behaviour of the derivatives of the fundamental solution, $\Pi^{(j)}(\mathbf{x},\mathbf{y})$, $j = 1,2$, for $|\mathbf{x}-\mathbf{y}| \to 0$. We will find that for the special choice $\tilde{\mu} = \mu\,(\mu+\lambda)/(3\mu+\lambda)$ and $\tilde{\lambda} = (2\mu+\lambda)(\mu+\lambda)/(3\mu+\lambda)$, these matrix functions become weakly singular. In this case, $\mathbf{P}$ is called the pseudo stress operator (Kupradze [35]).

To simplify the investigations, we introduce Kelvin's matrix, the matrix of fundamental solutions for elasto-static problems, i.e. boundary value problems involving the Navier equation with $\omega = 0$:

$$\Gamma_K(\mathbf{x},\mathbf{y}) := \frac{3\mu+\lambda}{4\pi\,\mu\,(2\mu+\lambda)}\,\log\frac{1}{|\mathbf{x}-\mathbf{y}|}\,\mathbf{I} + \frac{\mu+\lambda}{4\pi\,\mu\,(2\mu+\lambda)}\,\mathbf{J}(\mathbf{x}-\mathbf{y}),$$

for $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^2$, $\mathbf{x} \neq \mathbf{y}$, with

$$\mathbf{J}(\mathbf{z}) := \frac{\mathbf{z}\,\mathbf{z}^\top}{|\mathbf{z}|^2}.$$

As the following theorem shows, the matrix function Γ_K has the same singular behaviour as Γ itself.

Theorem 3.2 *The matrix function* Δ *defined by*

$$\Delta(\mathbf{x},\mathbf{y}) := \Gamma(\mathbf{x},\mathbf{y}) - \Gamma_K(\mathbf{x},\mathbf{y}), \qquad \mathbf{x} \neq \mathbf{y},$$

belongs to $[C^1(\mathbb{R}^2 \times \mathbb{R}^2)]^{2\times 2}$.

Proof: It is only necessary to show that Δ is well-defined and continuously differentiable for $\mathbf{x} = \mathbf{y}$. From (2.13) and asymptotic expansions for Φ_1, Φ_2 given in [34], we see that

$$\Delta(\mathbf{x},\mathbf{y}) = \tilde{\Phi}_1(|\mathbf{x}-\mathbf{y}|)\,\mathbf{I} + \tilde{\Phi}_2(|\mathbf{x}-\mathbf{y}|)\,\mathbf{J}(\mathbf{x}-\mathbf{y}),$$

with

$$\begin{aligned}\tilde{\Phi}_1(t) &= \beta_1\, t^2\, \log t + \gamma + \log t\, O(t^4) + O(t^2),\\ \tilde{\Phi}_2(t) &= \beta_2\, t^2\, \log t + \log t\, O(t^4) + O(t^2)\end{aligned}$$

as $t \to 0$, with complex constants β_1, β_2 and γ.

We further introduce the matrix function $\mathbf{H}^{(l)}$ by

$$\mathbf{H}^{(l)}(\mathbf{z}) := -2\, z_l\, \frac{\mathbf{z}\,\mathbf{z}^\top}{|\mathbf{z}|^4} + \frac{\mathbf{e}_l\,\mathbf{z}^\top + \mathbf{z}\,\mathbf{e}_l^\top}{|\mathbf{z}|^2}, \qquad l = 1, 2,$$

for $\mathbf{z} \in \mathbb{R}^2$, where $\mathbf{e}_l$ denotes the l-th cartesian unit coordinate vector. An easy calculation then shows

$$\begin{aligned}\frac{\partial}{\partial x_l}\,\Delta(\mathbf{x},\mathbf{y}) &= \tilde{\Phi}_1'(|\mathbf{x}-\mathbf{y}|)\,\frac{x_l - y_l}{|\mathbf{x}-\mathbf{y}|}\,\mathbf{I} + \tilde{\Phi}_2'(|\mathbf{x}-\mathbf{y}|)\,\frac{x_l - y_l}{|\mathbf{x}-\mathbf{y}|}\,\mathbf{J}(\mathbf{x}-\mathbf{y})\\ &\qquad + \tilde{\Phi}_2(|\mathbf{x}-\mathbf{y}|)\,\mathbf{H}^{(l)}(\mathbf{x}-\mathbf{y}).\end{aligned}$$

As

$$\begin{aligned}\tilde{\Phi}_1'(t) &= 3\beta_1\, t\, \log t + \log t\, O(t^3) + O(t),\\ \tilde{\Phi}_2'(t) &= 3\beta_2\, t\, \log t + \log t\, O(t^3) + O(t)\end{aligned}$$

as $t \to 0$, the assertion follows. ■

We will now apply the generalised stress operator $\mathbf{P}^{(\mathbf{x})}$ to the columns of Γ_K. Denote by Λ a curve of class $C^{1,1}$ in $\mathbb{R}^2$ and by $\mathbf{n}(\mathbf{x})$ its normal at $\mathbf{x} \in \Lambda$. For $\mathbf{x} \in \Lambda$, $\mathbf{y} \in \mathbb{R}^2$ and setting $r := |\mathbf{x}-\mathbf{y}|$ as well as $C := (3\mu+\lambda)/(4\pi\mu(2\mu+\lambda))$, we obtain

$$\Gamma_{K,jk}(\mathbf{x},\mathbf{y}) = C\left\{\delta_{jk}\, \log\frac{1}{r} + \frac{\mu+\lambda}{3\mu+\lambda}\,\frac{\partial r}{\partial x_j}\,\frac{\partial r}{\partial x_k}\right\}.$$

Thus, after some calculation,

$$\begin{aligned}\frac{\partial \Gamma_{K,jk}(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{x})} &= C\Big\{\delta_{jk}\,\frac{\partial}{\partial \mathbf{n}(\mathbf{x})}\,\log\frac{1}{r}+\frac{\mu+\lambda}{3\mu+\lambda}\sum_{l=1}^{2} n_l\,\frac{\partial}{\partial x_l}\left(\frac{\partial r}{\partial x_j}\,\frac{\partial r}{\partial x_k}\right)\Big\}\\ &= C\Big\{\frac{\partial}{\partial \mathbf{n}(\mathbf{x})}\,\log\frac{1}{r}\left(\delta_{jk}+2\,\frac{\mu+\lambda}{3\mu+\lambda}\,\frac{\partial r}{\partial x_j}\,\frac{\partial r}{\partial x_k}\right)\\ &\qquad -\frac{\mu+\lambda}{3\mu+\lambda}\left(n_j\,\frac{\partial}{\partial x_k}\,\log\frac{1}{r}+n_k\,\frac{\partial}{\partial x_j}\,\log\frac{1}{r}\right)\Big\}.\end{aligned}$$

Similarly, we obtain

$$\mathrm{div}_{\mathbf{x}}\Gamma_{K,\cdot k}=\frac{2\mu}{3\mu+\lambda}\,\frac{\partial}{\partial x_k}\,\log\frac{1}{r},$$

and thus

$$\left(\mathbf{n}(\mathbf{x})\,\mathrm{div}_{\mathbf{x}}\Gamma_{K,\cdot k}\right)_j=\frac{2\mu}{3\mu+\lambda}\,n_j\,\frac{\partial}{\partial x_k}\,\log\frac{1}{r}.$$

Finally, there holds

$$\begin{aligned}\left(\mathbf{n}^{\perp}(\mathbf{x})\,\mathrm{div}_{\mathbf{x}}^{\perp}\Gamma_{K,\cdot k}\right)_j &= \sum_{l=1}^{2} n_l\left(\frac{\partial \Gamma_{K,jk}(\mathbf{x},\mathbf{y})}{\partial x_l}-\frac{\partial \Gamma_{K,lk}(\mathbf{x},\mathbf{y})}{\partial x_j}\right)\\ &= 2C\frac{2\mu+\lambda}{3\mu+\lambda}\Big\{\delta_{jk}\,\frac{\partial}{\partial \mathbf{n}(\mathbf{x})}\,\log\frac{1}{r}-n_k\,\frac{\partial}{\partial x_j}\,\log\frac{1}{r}\Big\}.\end{aligned}$$

Combining these results yields

$$\begin{aligned}\left(\mathbf{P}^{(\mathbf{x})}(\Gamma_{K,\cdot k}(\mathbf{x},\mathbf{y}))\right)_j &= C\,\frac{\partial}{\partial \mathbf{n}(\mathbf{x})}\,\log\frac{1}{r}\Big((\mu+\tilde{\mu}-2\tilde{\mu}\,\frac{2\mu+\lambda}{3\mu+\lambda})\,\delta_{jk}\\ &\qquad +2\,(\mu+\tilde{\mu})\,\frac{\mu+\lambda}{3\mu+\lambda}\,\frac{\partial r}{\partial x_j}\,\frac{\partial r}{\partial x_k}\Big)\\ &\quad +C\,n_k\,\frac{\partial}{\partial x_j}\,\log\frac{1}{r}\left(2\tilde{\mu}\,\frac{2\mu+\lambda}{3\mu+\lambda}-(\mu+\tilde{\mu})\,\frac{\mu+\lambda}{3\mu+\lambda}\right)\\ &\quad +Cn_j\,\frac{\partial}{\partial x_k}\,\log\frac{1}{r}\left(\tilde{\lambda}\,\frac{2\mu}{3\mu+\lambda}-(\mu+\tilde{\mu})\,\frac{\mu+\lambda}{3\mu+\lambda}\right).\end{aligned}\tag{3.5}$$

We thus immediately obtain the following lemma:

Lemma 3.3 *Let Λ denote a curve of class $C^{1,1}$ in $\mathbb{R}^2$ and assume $\mathbf{x}\in\Lambda$, $\mathbf{y}\in\mathbb{R}^2$. For the choice $\tilde{\mu}=\mu\,(\mu+\lambda)/(3\mu+\lambda)$ and $\tilde{\lambda}=(2\mu+\lambda)(\mu+\lambda)/(3\mu+\lambda)$, there holds*

$$\left(\mathbf{P}^{(\mathbf{x})}(\Gamma_{K,\cdot k}(\mathbf{x},\mathbf{y}))\right)_j=\frac{1}{2\pi}\left(\frac{2\mu+\lambda}{3\mu+\lambda}\,\delta_{jk}+2\,\frac{\mu+\lambda}{3\mu+\lambda}\,\frac{\partial r}{\partial x_j}\,\frac{\partial r}{\partial x_k}\right)\frac{\partial}{\partial \mathbf{n}(\mathbf{x})}\,\log\frac{1}{r}.$$

Proof: Immediate from (3.5) by inserting the given expressions for $\tilde{\mu}$ and $\tilde{\lambda}$. ∎

Remark 3.4 *Let Λ denote a curve of class $C^{1,1}$ in $\mathbb{R}^2$ and assume $\mathbf{x}$, $\mathbf{y} \in \Lambda$. Then, as a consequence of Theorem 3.2 and Lemma 3.3 and also observing Theorem 2.9 (c), we have that the estimate (2.14) holds with Γ replaced by $\Pi^{(j)}$, $j = 1, 2$.*

3.3 Uniform Regularity Results for Elastic Potentials on Bounded Surfaces

It is the goal of this section to state regularity results and jump relations for elastic potentials defined on smooth, bounded, closed surfaces. These results are in principle well known [24, 35]. In our presentation here, we will rely heavily on the proofs given in [24], but the results will be generalised in two important aspects. Firstly, we shall consider boundary curves of class $C^{1,1}$ instead of those of class C^2. This extension is important for the solvability theory presented in Chapter 5 as this theory requires a compactness property of bounded families of such surfaces in a weak topology. This property would not be satisfied without additional equicontinuity assumptions by families of C^2 surfaces.

Secondly, we shall identify the properties of the boundary curves that determine the constants in the regularity estimates. We are thus able to formulate these results *uniformly* for classes of domains sharing these properties.

Let us address the generalisation to $C^{1,1}$ boundary curves first. For a detailed analysis of potentials defined on Lyapunov surfaces, see e.g. [29]. For a boundary ∂D of class $C^{1,1}$, the curvature $\kappa(\mathbf{x})$ can be defined for allmost all $\mathbf{x} \in \partial D$. Moreover, recalling the remarks on $C^{1,\alpha}$ domains in Section 1.3, there holds $\kappa \in L^\infty(\partial D)$ and $\|\kappa\|_{L^\infty(\partial D)} \leq H_1(D)$.

A careful review of the proof of the regularity estimates in [24] reveals that the assumption of a C^2 boundary is only used to obtain certain geometrical estimates through Taylor expansions up to second order. In the proof of the following lemma (Lemma 1.1 in [24]) we indicate how these results can be proved in the case of a boundary of class $C^{1,1}$.

Lemma 3.5 *For some numbers κ_0, δ, $M > 0$, assume $D \in \mathcal{D}_{1,\kappa_0,\delta,M}$. Then*

$$\begin{aligned} |\mathbf{n}(\mathbf{x}) \cdot (\mathbf{x} - \mathbf{y})| &\leq q\,|\mathbf{x} - \mathbf{y}|^2, \\ |\mathbf{n}(\mathbf{x}) - \mathbf{n}(\mathbf{y})| &\leq q\,|\mathbf{x} - \mathbf{y}| \end{aligned}$$

for all $\mathbf{x}$, $\mathbf{y} \in \partial D$, *with the constant* q *given by*

$$q = \max\left\{\sqrt{2}\kappa_0, 2(\min\{\delta, \frac{1}{2\kappa_0}\})^{-1}, (\min\{\delta, \frac{1}{2\kappa_0}\})^{-2}M\right\}.$$

Proof: Let ∂D be parametrised in terms of its arc length and denote by $\mathbf{t}(s)$ and $\mathbf{n}(s)$ the tangent and normal vectors at s, respectively, and by $\kappa(s)$ the curvature at s, for all s where it is defined. Then

$$\begin{aligned} \frac{\partial}{\partial s}|\mathbf{x}(s) - \mathbf{x}(s_0)|^2 &= 2\,\mathbf{t}(s)\cdot(\mathbf{x}(s) - \mathbf{x}(s_0)), \\ \frac{\partial^2}{\partial s^2}|\mathbf{x}(s) - \mathbf{x}(s_0)|^2 &= 2\,[1 - \kappa(s)\,\mathbf{n}(s)\cdot(\mathbf{x}(s) - \mathbf{x}(s_0))], \end{aligned}$$

where the second equation holds for almost all s. Thus

$$\begin{aligned} |\mathbf{x}(s) - \mathbf{x}(s_0)|^2 &= 2\int_{s_0}^{s}\int_{s_0}^{t}[1 - \kappa(t')\,\mathbf{n}(t')\cdot(\mathbf{x}(t') - \mathbf{x}(s_0)]\,dt'\,dt, \\ \mathbf{n}(s_0)\cdot(\mathbf{x}(s) - \mathbf{x}(s_0)) &= -\int_{s_0}^{s}\int_{s_0}^{t}\kappa(t')\,\mathbf{n}(t')\cdot\mathbf{n}(s_0)\,dt'\,dt. \end{aligned}$$

First suppose that $|\mathbf{x}(s) - \mathbf{x}(s_0)| \leq \min\{\delta, (2\kappa_0)^{-1}\}$. Then an easy estimate of the double integral in the first equation yields

$$|\mathbf{x}(s) - \mathbf{x}(s_0)|^2 \geq \frac{1}{2}\,(s - s_0)^2,$$

and consequently, from the second equation,

$$|\mathbf{n}(s_0)\cdot(\mathbf{x}(s) - \mathbf{x}(s_0))| \leq \kappa_0\,|\mathbf{x}(s) - \mathbf{x}(s_0)|^2.$$

For $|\mathbf{x}(s) - \mathbf{x}(s_0)| > \min\{\delta, (2\kappa_0)^{-1}\}$, there trivially holds

$$|\mathbf{n}(s_0)\cdot(\mathbf{x}(s) - \mathbf{x}(s_0))| \leq (\min\{\delta, (2\kappa_0)^{-1}\})^{-2}M\,|\mathbf{x}(s) - \mathbf{x}(s_0)|^2.$$

Similarly, we have

$$\mathbf{n}(s) - \mathbf{n}(s_0) = \int_{s_0}^{s}\kappa(t)\,\mathbf{t}(t)\,dt.$$

The same reasoning as above now yields

$$|\mathbf{n}(s) - \mathbf{n}(s_0)| \leq \max\left\{\sqrt{2}\,\kappa_0, 2\,(\min\{\delta, (2\kappa_0^{-1}\})^{-1}\right\}|\mathbf{x}(s) - \mathbf{x}(s_0)|.$$

■

For the following arguments, let κ_0, δ, $M > 0$. For any $D \in \mathcal{D}_{1,\kappa_0,\delta,M}$ and $\phi \in [C(\partial D)]^2$, we define the elastic single-layer potential $\mathbf{v}_D$ and double-layer potential $\mathbf{w}_D$ by (3.3) and (3.4), respectively. Subsequently, let a superscript $^-$ denote vector fields defined in D and a superscript $^+$ vector fields defined in $\mathbb{R}^2 \setminus \bar{D}$.

The following theorem states regularity results for the elastic single layer potential.

Theorem 3.6 *Assume $D \in \mathcal{D}_{1,\kappa_0,\delta,M}$.*

(a) *For $\phi \in [C(\partial D)]^2$ and $\alpha \in (0,1)$, there hold $\mathbf{v}_D \in [C^{0,\alpha}(\mathbb{R}^2)]^2$ and*

$$\|\mathbf{v}_D\|_{0,\alpha;\mathbb{R}^2} \leq C\,\|\phi\|_{\infty;\partial D},$$

where the constant C depends only on α, κ_0, δ and M.

(b) *For $\phi \in [C^{0,\alpha}(\partial D)]^2$, $\alpha \in (0,1)$, the first order derivatives of $\mathbf{v}_D^{\pm}$ in $\mathbb{R}^2 \setminus \bar{D}$ and in D have $C^{0,\alpha}$-extensions to $\mathbb{R}^2 \setminus D$ and $\bar{D}$ respectively. Furthermore, there holds*

$$\|\mathbf{v}_D^-\|_{1,\alpha;\bar{D}},\ \|\mathbf{v}_D^+\|_{1,\alpha;\mathbb{R}^2\setminus D} \leq C\|\phi\|_{0,\alpha;\partial D},$$

where the constant C depends only on α, κ_0, δ and M. For $\tilde{\mu} = \mu\,(\mu+\lambda)/(3\mu+\lambda)$ and $\tilde{\lambda} = (2\mu+\lambda)(\mu+\lambda)/(3\mu+\lambda)$, there holds

$$\mathbf{P}\mathbf{v}_D^{\pm}(\mathbf{x}) = \mp\frac{1}{2}\phi(\mathbf{x}) + \int_{\partial D} \Pi^{(1)}(\mathbf{x},\mathbf{y})\,\phi(\mathbf{y})\,ds(\mathbf{y}), \qquad \mathbf{x} \in \partial D, \tag{3.6}$$

where the integral exists as an improper integral.

(c) *For $\phi \in [C(\partial D)]^2$ and $\alpha \in (0,1)$, there holds*

$$\int_{\partial D} \Gamma(\cdot,\mathbf{y})\,\phi(\mathbf{y})\,ds(\mathbf{y})\ \in [C^{0,\alpha}(\partial D)]^2,$$

where the integral exists as an improper integral. Furthermore,

$$\left\|\int_{\partial D} \Gamma(\cdot,\mathbf{y})\,\phi(\mathbf{y})\,ds(\mathbf{y})\right\|_{0,\alpha;\partial D} \leq C\,\|\phi\|_{\infty;\partial D},$$

where the constant C depends only on α, κ_0, δ and M.

Proof: The Theorem is proved in the same way as Theorem 2.19 and Theorem 2.21 in [24].

For the dependence of the constant C on the parameters, a review of the proof in [24] shows that all constants depend on κ_0, δ and M in a similar way as the constant q in Lemma 3.5.

Applying the arguments of [24] it follows that (3.6) holds for all $\tilde{\lambda}$ and $\tilde{\mu}$, with the integral defined in a Cauchy principal value sense. That the integral exists as an improper integral for $\tilde{\mu} = \mu\,(\mu+\lambda)/(3\mu+\lambda)$ and $\tilde{\lambda} = (2\mu+\lambda)(\mu+\lambda)/(3\mu+\lambda)$ follows from Theorem 3.2 and Lemma 3.3.

Note that (c) is implied by (a). ∎

The corresponding theorem for the double-layer potential is as follows:

Theorem 3.7 *Assume $D \in \mathcal{D}_{1,\kappa_0,\delta,M}$.*

(a) For $\phi \in [C(\partial D)]^2$, $\tilde{\mu} = \mu\,(\mu+\lambda)/(3\mu+\lambda)$ and $\tilde{\lambda} = (2\mu+\lambda)(\mu+\lambda)/(3\mu+\lambda)$, the double layer potential $\mathbf{w}_D^{\pm}$ in $\mathbb{R}^2 \setminus \bar{D}$ and in D has continuous extensions to $\mathbb{R}^2 \setminus D$ and $\bar{D}$ respectively. Furthermore, there holds

$$\|\mathbf{w}_D^-\|_{\infty;\bar{D}},\ \|\mathbf{w}_D^+\|_{\infty;\mathbb{R}^2\setminus D} \leq C\,\|\phi\|_{\infty;\partial D},$$

where the constant C depends only on κ_0, δ and M, and

$$\mathbf{w}_D^{\pm}(\mathbf{x}) = \pm\frac{1}{2}\phi(\mathbf{x}) + \int_{\partial D} \Pi^{(2)}(\mathbf{x},\mathbf{y})\,\phi(\mathbf{y})\,ds(\mathbf{y}), \qquad \mathbf{x} \in \partial D,$$

where the integral exists as an improper integral.

(b) For $\phi \in [C(\partial D)]^2$, $\tilde{\mu} = \mu\,(\mu+\lambda)/(3\mu+\lambda)$ and $\tilde{\lambda} = (2\mu+\lambda)(\mu+\lambda)/(3\mu+\lambda)$ and $\alpha \in (0,1)$, there holds

$$\int_{\partial D} \Pi^{(j)}(\cdot,\mathbf{y})\,\phi(\mathbf{y})\,ds(\mathbf{y}) \ \in [C^{0,\alpha}(\partial D)]^2, \qquad j = 1,2,$$

where the integral exists as an improper integral. Furthermore,

$$\left\| \int_{\partial D} \Pi^{(j)}(\cdot,\mathbf{y})\,\phi(\mathbf{y})\,ds(\mathbf{y}) \right\|_{0,\alpha;\partial D} \leq C\,\|\phi\|_{\infty;\partial D}, \qquad j = 1,2,$$

where the constant C depends only on α, κ_0, δ and M.

Proof: Observing also Lemma 3.3, the assertion is deduced analogously to [24, Theorem 2.20], but making use of the fact that the integral kernel is weakly singular. Also note Theorem 2.9 (c) to show the assertion on the integral in the formula for $\mathbf{w}_D^{\pm}(\mathbf{x})$, $\mathbf{x} \in \partial D$. ■

It is also possible to study the regularity of the elastic surface potentials if the density is in a Sobolev space of fractional order. Such investigations have e.g. been carried out in [25]. We will only make use of the following result:

Remark 3.8 *Assume $D \subset \mathbb{R}^2$ to be of class $C^{1,1}$, $\tilde{\mu} = \mu\,(\mu+\lambda)/(3\mu+\lambda)$ and $\tilde{\lambda} = (2\mu+\lambda)(\mu+\lambda)/(3\mu+\lambda)$ and $\phi \in [H^{1/2}(\partial D)]^2$. Then there holds $\mathbf{v}_D$, $\mathbf{w}_D \in [H^1_{loc}(\mathbb{R}^2 \setminus \partial D)]^2$.*

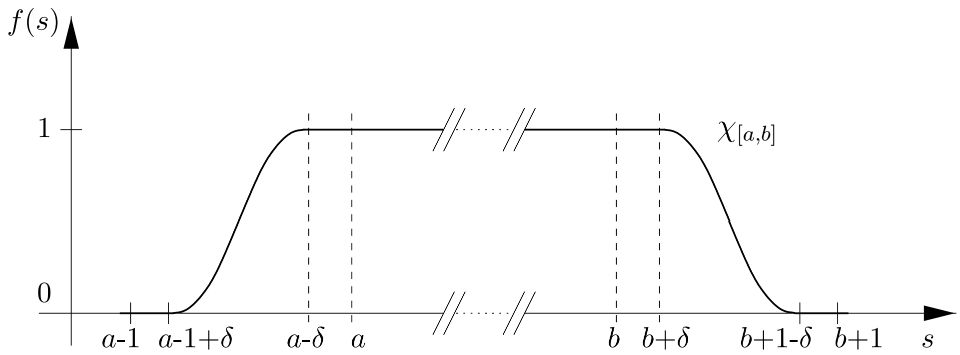

Figure 3.1: The function $\chi_{[a,b]}$

3.4 The Regularity of Elastic Potentials defined on Rough Surfaces

To obtain similar regularity results for elastic potentials defined on an unbounded surface, the kernel functions will be separated into localised singular parts and the remainders which are smooth. The regularity estimates obtained will be uniform with respect to certain classes of boundaries.

To this end, fix $h \in \mathbb{R}$ and also $c > h$ and $M > 0$. Recalling the definition of the set $B_{1,c,M}$ in Section 1.3, we will show the regularity of the potentials uniformly with respect to functions $f \in B_{1,c,M}$.

Also fix $H > M \geq \sup_{f \in B_{1,c,M}} \|f\|_{\infty;\mathbb{R}}$ and introduce the domains

$$V_n := \{\mathbf{x} \in U_h : n-1 < x_1 < n+1, x_2 < H\}, \qquad n \in \mathbb{Z}.$$

It now follows that for each $\mathbf{x}$, $\mathbf{y} \in U_h \setminus \overline{U_H}$ there either exists $n \in \mathbb{Z}$ such that $\mathbf{x}$, $\mathbf{y} \in V_n$ or $|\mathbf{x} - \mathbf{y}| \geq 1$ must hold.

We also introduce a C^∞ function χ with the following properties: $\chi(s) = 0$ for $s < \varepsilon$, $\chi(s) = 1$ for $s \geq 1 - \varepsilon$ for some ε, $1/2 > \varepsilon > 0$. For $a < b$, we now define (see also Figure 3.1)

$$\chi_{[a,b]}(s) := \begin{cases} 1, & a \leq s \leq b, \\ \chi(s-a+1), & a-1 < s < a, \\ \chi(b-s+1), & b < s < b+1, \\ 0 & \text{otherwise} \end{cases}.$$

Then $\chi_{[a,b]} \in C_0^\infty(\mathbb{R})$.

In the following arguments, we will indicate dependence of a function or set on the boundary function $f \in B_{1,c,M}$ by a sub- or superscript f. Recalling definition (3.1) of the elastic single layer potential on a rough surface, we define for $f \in B_{1,c,M}$ and $\phi \in [BC(S_f)]^2$

$$\mathbf{v}^f_{1,n}(\mathbf{x}) := \int_{S_f} \big(\Gamma_{D,h}(\mathbf{x},\mathbf{y}) - \chi_{[n-1,n+1]}(y_1)\,\Gamma(\mathbf{x},\mathbf{y})\big)\ \phi(\mathbf{y})\,ds(\mathbf{y}), \qquad \mathbf{x} \in V_n \setminus S_f,$$

and

$$\mathbf{v}^f_{2,n}(\mathbf{x}) := \int_{S_f} \Gamma(\mathbf{x},\mathbf{y})\,\chi_{[n-1,n+1]}(y_1)\,\phi(\mathbf{y})\,ds(\mathbf{y}), \qquad \mathbf{x} \in V_n \setminus S_f.$$

On $V_n \setminus S_f$ there obviously holds $\mathbf{v}^f = \mathbf{v}^f_{1,n} + \mathbf{v}^f_{2,n}$. We now analyse the regularity of these two vector fields.

Lemma 3.9 *For $f \in B_{1,c,M}$ and $\alpha \in (0,1)$, there holds $\mathbf{v}^f_{1,n} \in C^{0,\alpha}(V_n)$ and*

$$\|\mathbf{v}^f_{1,n}\|_{0,\alpha;V_n} \le C\|\phi\|_{\infty;S_f},$$

where the constant C only depends on α, c, M, h and H.

Proof: For $\mathbf{x} \in V_n$, $\mathbf{y} \in S_f$, set

$$K_{jk}(\mathbf{x},\mathbf{y}) := \Gamma_{D,h,jk}(\mathbf{x},\mathbf{y}) - \chi_{[n-1,n+1]}(y_1)\,\Gamma_{jk}(\mathbf{x},\mathbf{y}), \qquad j,k = 1,2.$$

From Theorems 2.10 and 2.13 as well as Theorem 2.16 we know K_{jk} is continuously differentiable in $V_n \times S_f$ and that there exists a function $\kappa \in L^1(\mathbb{R}) \cap BC(\mathbb{R})$, dependent only on c, M, h and H such that

$$\left.\begin{array}{c} |K_{jk}(\mathbf{x},\mathbf{y})| \\ |\mathrm{grad}_{\mathbf{x}} K_{jk}(\mathbf{x},\mathbf{y})| \end{array}\right\} \le \kappa(x_1 - y_1), \qquad \mathbf{x} \in V_n, \mathbf{y} \in S_f. \tag{3.7}$$

Thus, we immediately obtain, for $\mathbf{x} \in V_n$,

$$|\mathbf{v}^f_{1,n}(\mathbf{x})| \le 4\,(1+M)\,\|\kappa\|_{L^1(\mathbb{R})}\,\|\phi\|_{\infty;S_f}, \qquad \mathbf{x} \in V_n. \tag{3.8}$$

Now, for some $\varepsilon > 0$, set

$$S_{\mathbf{x},\varepsilon} := \{\mathbf{y} \in S_f : |x_1 - y_1| < \varepsilon\}, \qquad \mathbf{x} \in V_n.$$

Then, letting $\mathbf{x}$, $\mathbf{x}' \in V_n$ and assuming $|\mathbf{x} - \mathbf{x}'| < \varepsilon$, from (3.7) and the Mean Value Theorem it follows that

$$\begin{aligned} &\left| \int_{S_{\mathbf{x},2\varepsilon}} (K(\mathbf{x},\mathbf{y}) - K(\mathbf{x}',\mathbf{y}))\,\phi(\mathbf{y})\,ds(\mathbf{y}) \right| \\ &\quad \le \ 16\,(1+M)\,\|\kappa\|_{\infty;\mathbb{R}}\,\varepsilon^{2-\alpha}\,\|\phi\|_{\infty;S_f}\,|\mathbf{x} - \mathbf{x}'|^{\alpha}. \end{aligned} \tag{3.9}$$

On the other hand, we also have the estimate

$$\left|\int_{S_f\setminus S_{\mathbf{x},2\varepsilon}} (K(\mathbf{x},\mathbf{y}) - K(\mathbf{x}',\mathbf{y}))\,\phi(\mathbf{y})\,ds(\mathbf{y})\right| \leq 4\,(1+M)\,\|\kappa\|_{L^1(\mathbb{R})}\,\varepsilon^{1-\alpha}\,\|\phi\|_{\infty;S_f}\,|\mathbf{x}-\mathbf{x}'|^{\alpha}. \tag{3.10}$$

Combining (3.8)–(3.10) yields the assertion for $|\mathbf{x}-\mathbf{x}'| < \varepsilon$. In the case $|\mathbf{x}-\mathbf{x}'| > \varepsilon$, from (3.8) we trivially obtain

$$|\mathbf{v}^f_{1,n}(\mathbf{x}) - \mathbf{v}^f_{1,n}(\mathbf{x}')| \leq 8\,(1+M)\,\varepsilon^{-\alpha}\,\|\kappa\|_{L^1(\mathbb{R})}\,\|\phi\|_{\infty;S_f}\,|\mathbf{x}-\mathbf{x}'|^{\alpha}.$$

■

Lemma 3.10 *For $f \in B_{1,c,M}$, $\alpha \in (0,1)$, there holds $\mathbf{v}^f_{2,n} \in C^{0,\alpha}(V_n)$ and*

$$\|\mathbf{v}^f_{2,n}\|_{0,\alpha;V_n} \leq C\|\phi\|_{\infty;S_f},$$

where the constant C only depends on α, c, M, h and H.

Proof: We define domains D^f_n, $f \in B_{1,c,M}$, in a manner very similar to the construction described in Chapter 2, just before Theorem 2.7. Choosing $\rho > 0$, we define

$$\chi_n(s) := \begin{cases} \chi(s-n+3), & s < n-2, \\ 1, & n-2 \leq s \leq n+2, \\ \chi(n+3-s), & n+2 < s, \end{cases}$$

where χ is the same cut-off function used earlier in this section, and set

$$\tilde{f}(s) := \chi_n(s)\,f(s) + (1-\chi_n(s))\,(M+\rho).$$

A closed $C^{1,1}$ boundary curve ∂D^f_n will now be constructed as indicated below and illustrated in Figure 3.2.

- between the points $(n-3,\tilde{f}(n-3))^\top$ and $(n+3,\tilde{f}(n+3))^\top$, ∂D^f_n is identical to $\tilde{S}_f := \{\mathbf{x} \in \mathbb{R}^2 : x_2 = \tilde{f}(x_1)\}$,
- outside this section, ∂D^f_n is continued by two half circles with radius ρ,
- ∂D^f_n is closed by a straight line connecting the two half circles.

The domain D^f_n is thus a bounded, simply connected domain of class $C^{1,1}$. Moreover, there are constants κ_0, δ and $\tilde{M}$ only dependent on c, M and ρ such that $D^f_n \subset \mathcal{D}_{1,\kappa_0,\delta,\tilde{M}}$ for all $n \in \mathbb{Z}$ and all $f \in B_{1,c,M}$. Also, between $(n-2, f(n-2))$ and $(n+2, f(n+2))$, ∂D^f_n is identical to S_f.

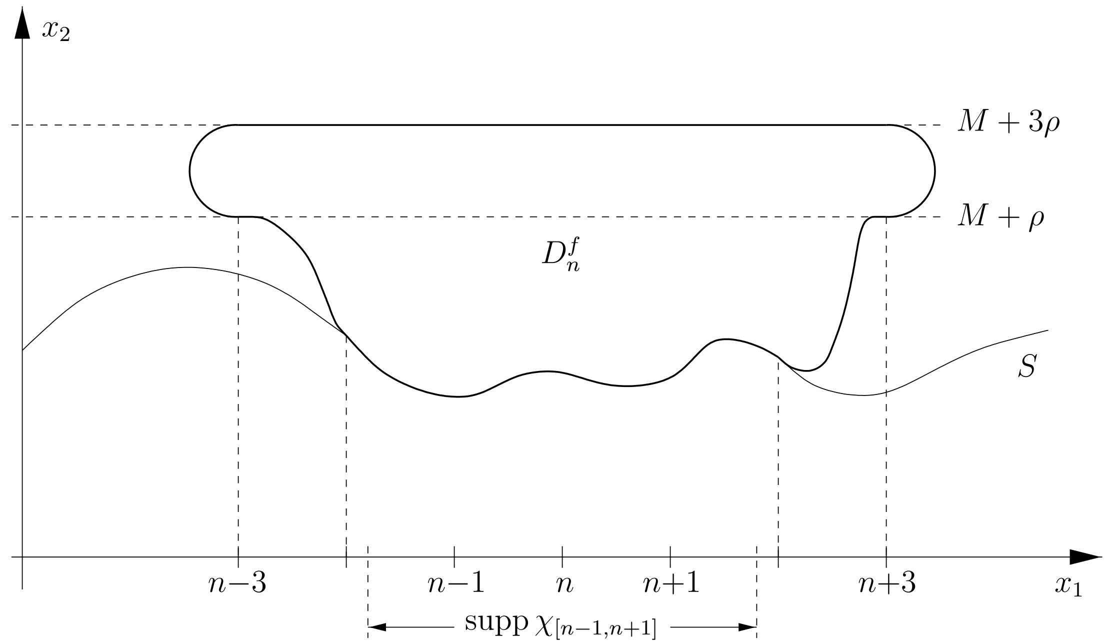

Figure 3.2: Construction of the domain D_n^f

As supp$\chi_{[n-1,n+1]} \subset\subset (n-2, n+2)$ we can define the density $\psi \in [BC(\partial D_n^f)]^2$ by

$$\psi(\mathbf{x}) := \begin{cases} \chi_{[n-1,n+1]}(x_1)\,\phi(\mathbf{x}), & \mathbf{x} \in \partial D_n^f \cap S_f, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\mathbf{v}_{2,n}^f(\mathbf{x}) = \int_{\partial D_n^f} \Gamma(\mathbf{x},\mathbf{y})\,\psi(\mathbf{y})\,ds(\mathbf{y}).$$

An application of Theorem 3.6 (a) now completes the proof. ■

The two previous lemmas are the building blocks for the proof of the next theorem in which the regularity results for the elastic single layer potential on a rough surface are stated. In its formulation, and also in the subsequent arguments in this chapter, let a superscript $^+$ vector fields defined in Ω_f and a superscript $^-$ denote vector fields defined in $U_h \setminus \overline{\Omega_f}$.

Theorem 3.11 *Assume $f \in B_{1,c,M}$.*

(a) For $\phi \in [BC(S_f)]^2$ and $\alpha \in (0,1)$, there holds $\mathbf{v}_f \in [\mathcal{V}^{0,\alpha}(U_h)]^2$ and

$$\|\mathbf{v}_f\|_{0,\alpha;U_h\setminus\overline{U_H}} \le C\,\|\phi\|_{\infty;S_f},$$

where the constant C depends only on α, c, M, h and H.

(b) For $\phi \in [C^{0,\alpha}(S_f)]^2$*,* $\alpha \in (0,1)$*, the first order derivatives of* $\mathbf{v}_f^{\pm}$ *in* Ω_f *and in* $U_h \setminus \overline{\Omega_f}$ *have* $C^{0,\alpha}$*-extensions to* $\overline{\Omega_f}$ *and* $U_h \setminus \Omega_f$ *respectively. Furthermore, there holds*

$$\|\mathbf{v}_f^{-}\|_{1,\alpha;U_h\setminus\Omega_f}, \ \|\mathbf{v}_f^{+}\|_{1,\alpha;\overline{\Omega_f}\setminus\overline{U_H}} \le C\,\|\phi\|_{0,\alpha;S_f},$$

where the constant C *depends only on* α*,* c*,* M*,* h *and* H*. For* $\tilde{\mu} = \mu(\mu+\lambda)/(3\mu+\lambda)$ *and* $\tilde{\lambda} = (2\mu+\lambda)(\mu+\lambda)/(3\mu+\lambda)$*, there holds*

$$\mathbf{P}\mathbf{v}_f^{\pm}(\mathbf{x}) = \mp\frac{1}{2}\phi(\mathbf{x}) + \int_{S_f} \Pi^{(1)}_{D,h}(\mathbf{x},\mathbf{y})\,\phi(\mathbf{y})\,ds(\mathbf{y}), \qquad \mathbf{x} \in S_f,$$

where the integral exists as an improper integral.

(c) For $\phi \in [BC(S_f)]^2$ *and* $\alpha \in (0,1)$*, there holds*

$$\int_{S_f} \Gamma_{D,h}(\cdot,\mathbf{y})\,\phi(\mathbf{y})\,ds(\mathbf{y}) \ \in [C^{0,\alpha}(S_f)]^2,$$

where the integral exists as an improper integral. Furthermore,

$$\left\| \int_{S_f} \Gamma_{D,h}(\cdot,\mathbf{y})\,\phi(\mathbf{y})\,ds(\mathbf{y}) \right\|_{0,\alpha;S_f} \le C\,\|\phi\|_{\infty;S_f},$$

where the constant C *depends only on* α*,* c*,* M *and* h*.*

Proof: We immediately conclude from Lemmas 3.9 and 3.10 that $\mathbf{v}_f \in C(U_h)$ and that

$$\|\mathbf{v}_f\|_{\infty;U_h\setminus\overline{U_H}} \le C\|\phi\|_{\infty;S_f} \tag{3.11}$$

where the constant C only depends on α, c, M, h and H. For $\mathbf{x}$, $\mathbf{y} \in U_h \setminus \overline{U_H}$, there either exists $n \in \mathbb{Z}$ with $\mathbf{x}$, $\mathbf{y} \in V_n$ or $|\mathbf{x}-\mathbf{y}| \ge 1$. In the second case, we trivially obtain from (3.11) that

$$|\mathbf{v}_f(\mathbf{x}) - \mathbf{v}_f(\mathbf{y})| \le C\|\phi\|_{\infty;S_f}\,|\mathbf{x}-\mathbf{y}|^{\alpha}.$$

In the first case, however, the same estimate follows from Lemmas 3.9 and 3.10. This concludes the proof of part (a).

As the kernel function in the definition of $\mathbf{v}^f_{1,n}$ is infinitely smooth, we can exchange the order of integration and differentiation. Thus, in a way directly analogous to the proof of Lemma 3.9 we obtain the same estimate for any first derivative of $\mathbf{v}^f_{1,n}$. To see (b), we now proceed as in the proof of Lemma 3.10, only applying Theorem 3.6 (b), and use the same arguments as for part (a).

Part (c) is an immediate consequence of part (a). ∎

The same arguments as for Theorem 3.11 can be applied to prove regularity for the double-layer potential. We only have to employ Theorem 3.7 to obtain the following theorem.

Theorem 3.12 *Assume* $f \in B_{1,c,M}$.

(a) For $\phi \in [BC(S_f)]^2$, $\tilde{\mu} = \mu\,(\mu+\lambda)/(3\mu+\lambda)$ *and* $\tilde{\lambda} = (2\mu+\lambda)(\mu+\lambda)/(3\mu+\lambda)$, *the double layer potential* $\mathbf{w}_f^{\pm}$ *in* Ω_f *and in* $U_h \setminus \overline{\Omega_f}$ *has continuous extensions to* $\overline{\Omega_f}$ *and* $U_h \setminus \Omega_f$ *respectively. Furthermore, there holds*

$$\|\mathbf{w}_f^-\|_{\infty;U_h\setminus\Omega_f},\ \|\mathbf{w}_f^+\|_{\infty;\overline{\Omega_f}} \le C\|\phi\|_{\infty;S_f},$$

where the constant C *depends only on* c, M, h *and* H, *and*

$$\mathbf{w}_f^{\pm}(\mathbf{x}) = \pm\frac{1}{2}\phi(\mathbf{x}) + \int_{S_f} \Pi_{D,h}^{(2)}(\mathbf{x},\mathbf{y})\,\phi(\mathbf{y})\,ds(\mathbf{y}), \qquad \mathbf{x} \in S_f,$$

where the integral exists as an improper integral.

(b) For $\phi \in [BC(S_f)]^2$, $\tilde{\mu} = \mu\,(\mu+\lambda)/(3\mu+\lambda)$, $\tilde{\lambda} = (2\mu+\lambda)(\mu+\lambda)/(3\mu+\lambda)$ *and* $\alpha \in (0,1)$, *there holds*

$$\int_{S_f} \Pi_{D,h}^{(j)}(\cdot,\mathbf{y})\,\phi(\mathbf{y})\,ds(\mathbf{y})\ \in [C^{0,\alpha}(S_f)]^2, \qquad j = 1,2,$$

where the integral exists as an improper integral. Furthermore,

$$\left\|\int_{S_f} \Pi_{D,h}^{(j)}(\cdot,\mathbf{y})\,\phi(\mathbf{y})\,ds(\mathbf{y})\right\|_{0,\alpha;S_f} \le C\,\|\phi\|_{\infty;S}, \qquad j = 1,2,$$

where the constant C *depends only on* α, c, M *and* h.

For certain arguments, it will also be necessary to study the elastic layer potentials in the Sobolev space $[H^1_{loc}(U_h)]^2$. Making use of Remark 3.8 and using similar arguments as in Lemmas 3.9 and 3.10, we obtain the following result:

Remark 3.13 *Assume* $\tilde{\mu} = \mu\,(\mu+\lambda)/(3\mu+\lambda)$ *and* $\tilde{\lambda} = (2\mu+\lambda)(\mu+\lambda)/(3\mu+\lambda)$ *and* $\phi \in [H^{1/2}_{loc}(S)]^2$. *Then there holds* $\mathbf{v}$, $\mathbf{w} \in [H^1_{loc}(U_h \setminus S)]^2$.

Chapter 4

Radiation Conditions and Uniqueness

In this chapter, the rough surface scattering problem will be formulated mathematically as a boundary value problem in the domain Ω. To ensure well-posedness of this boundary value problem, a *radiation condition* has to be included in the formulation. We will thus begin by investigating some of the radiation conditions that have been used in elastic wave scattering problems. We then proceed to define a new radiation condition, termed the *upward propagating radiation condition* (UPRC) and analyse its properties, in particular how it generalises some of the more conventional conditions. We will then present the formulation of the scattering problem as a boundary value problem and, eventually, prove that this problem admits at most one solution.

4.1 Radiation Conditions for Elastic Waves

From the perspective of physics, it is clear that the *scattered field* ought to be made up of waves travelling along or away from the scattering obstacle. When formulating the problem as a boundary value problem, uniqueness of solution can only be ensured if a mathematical characterisation of such fields is included in the formulation. Such a characterisation is termed a *radiation condition*.

Probably the best known radiation condition in scattering theory was that introduced by A. Sommerfeld in his Habilitation thesis [44] in 1896. A modern presentation of this condition and its role for scattering problems involving bounded obstacles is given by Colton/Kress [23]. The formulation of Sommerfeld's radiation condition for the elastic wave case is due to Kupradze [35], and a modern version of this formulation is given here:

Definition 4.1 *Let $D \subset \mathbb{R}^2$ be a bounded domain. A solution $\mathbf{u} \in [C^2(\mathbb{R}^2 \setminus \bar{D})]^2$ to the Navier equation in the exterior of D is said to satisfy* Kupradze's radiation condition *if*

$$\frac{\partial \mathbf{u}_p}{\partial r} - ik_p\mathbf{u}_p = o(r^{-1/2}) \qquad \text{and} \qquad \frac{\partial \mathbf{u}_s}{\partial r} - ik_s\mathbf{u}_s = o(r^{-1/2}) \tag{4.1}$$

uniformly in $\mathbf{x}/r$ as $r := |\mathbf{x}| \to \infty$.

In the case of scattering problems involving an effectively unbounded scatterer, it is not clear that (4.1) necessarily implies a specific decay rate for $\mathbf{u}(\mathbf{x})$ as $|\mathbf{x}| \to \infty$. Thus, for scattering problems of this type, we introduce the following notion of a radiating wave:

Definition 4.2 *Let $H \in \mathbb{R}$. A solution $\mathbf{u} \in [C^2(U_H)]^2$ to the Navier equation is said to be* radiating *if*

$$\mathbf{u}_p = O(r^{-1/2}), \qquad \frac{\partial \mathbf{u}_p}{\partial r} - ik_p\mathbf{u}_p = o(r^{-1/2})$$

and

$$\mathbf{u}_s = O(r^{-1/2}), \qquad \frac{\partial \mathbf{u}_s}{\partial r} - ik_s\mathbf{u}_s = o(r^{-1/2})$$

uniformly in $\mathbf{x}/r$ for $\mathbf{x} \in U_H$ as $r := |\mathbf{x}| \to \infty$.

Remark 4.3 *A vector field satisfying Kupradze's radiation condition in the sense of Definition 4.1 for some $D \subset \mathbb{R}^2$ is also radiating in the sense of Definition 4.2 for any H such that $D \cap U_H = \emptyset$ (see, e.g., formula (3.63) in [23]).*

As is to be expected, Γ also satisfies Kupradze's radiation condition:

Theorem 4.4 *The columns of the matrix functions $\Gamma(\cdot, \mathbf{y})$ and $\Pi^{(2)}(\cdot, \mathbf{y})$ as well as the rows of $\Gamma(\mathbf{x}, \cdot)$ and $\Pi^{(1)}(\mathbf{x}, \cdot)$, $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^2$, satisfy Kupradze's radiation condition.*

Proof: For Γ, the assertion is proved as in [35] for the three dimensional case. For $\Pi^{(1)}$, $\Pi^{(2)}$, it can be seen by an application of Lemma 2.5 together with the corresponding result for Γ. ■

In Section 2.4 when deriving the matrix of fundamental solutions $\Gamma_{D,h}$, it was formulated as a requirement that the columns of the matrix function $\mathbf{U}$ represent wave fields propagating away from T_h. We will now make this statement mathematically precise, by showing they represent radiating solutions to the Navier equation:

Theorem 4.5 *For* $\mathbf{x}$, $\mathbf{y} \in U_h$ *and* $H > \max\{x_2, y_2\}$, *the columns of the matrix function* $\Gamma_{D,h}(\cdot, \mathbf{y})$ *and the rows of the matrix function* $\Gamma_{D,h}(\mathbf{x}, \cdot)$ *are radiating solutions to the Navier equation in* U_H.

Proof: As Γ is symmetric, the assertion for the first two terms in (2.22) is proved in Theorem 4.4 together with Remark 4.3. Thus, it suffices to show the assertion for $\mathbf{U}$.

Observe that the terms in (2.27) involving M_p represent the longitudinal and the terms involving M_s the transversal part of $\mathbf{U}$; these will be denoted by $\mathbf{U}^{(p)}$ and $\mathbf{U}^{(s)}$, respectively. For fixed $\mathbf{y}$, an entry $\mathbf{U}^{(p)}_{jk}(\cdot, \mathbf{y})$, $j, k = 1, 2$, satisfies the scalar Helmholtz equation

$$\Delta_{\mathbf{z}} \, \mathbf{U}^{(p)}_{jk}(\mathbf{z}, \mathbf{y}) + k_p^2 \, \mathbf{U}^{(p)}_{jk}(\mathbf{z}, \mathbf{y}) = 0, \qquad \mathbf{z} \in U_h,$$

and the boundary condition

$$\mathbf{U}^{(p)}_{jk}(\mathbf{z}, \mathbf{y}) = g(\mathbf{z}) := -\Gamma_{jk}(\mathbf{z}, \mathbf{y}) + \Gamma_{jk}(\mathbf{z}, \mathbf{y}'_h) - \mathbf{U}^{(s)}_{jk}(\mathbf{z}, \mathbf{y}), \quad \mathbf{z} \in T_h.$$

From (2.27) we see, using arguments presented in [11], that $\mathbf{U}^{(p)}_{jk}$ satisfies the *upward propagating radiation condition* of [11, 19], given here as Definition 4.8 and, more precisely, that

$$\mathbf{U}^{(p)}_{jk}(\mathbf{z}, \mathbf{y}) = 2 \int_{T_h} \frac{\partial \Phi}{\partial \tilde{z}_2}(\mathbf{z}, \tilde{\mathbf{z}}) \, g(\tilde{\mathbf{z}}) \, ds(\tilde{\mathbf{z}}), \qquad \mathbf{z} \in U_h,$$

where $\Phi(\mathbf{z}, \tilde{\mathbf{z}}) = i/4 \, H_0^{(1)}(k_p|\mathbf{z} - \tilde{\mathbf{z}}|)$. Reviewing the proof of Theorem 2.13, we see that $g(\mathbf{z}) = O(|z_1|^{-3/2})$ as $|z_1| \to \infty$. We can thus use the argument presented in [14, Section 5] to conclude

$$|\mathbf{U}^{(p)}_{jk}(\mathbf{z}, \mathbf{y})| \leq C(1 + z_2 - h)(1 + r)^{-3/2}$$

and

$$\frac{\partial \mathbf{U}^{(p)}_{jk}}{\partial r}(\mathbf{z}, \mathbf{y}) - ik_p \mathbf{U}^{(p)_{jk}}(\mathbf{z}, \mathbf{y}) = o(r^{-1/2}),$$

where the derivative can be taken with respect to either $\mathbf{z}$ or $\mathbf{y}$. The same argument can be applied to $\mathbf{U}^{(s)}(\cdot, \mathbf{y})$ and the proof is now completed by recalling Lemma 2.15. ∎

Corollary 4.6 *For* $\mathbf{x}$, $\mathbf{y} \in U_h$, $H > \max\{x_2, y_2\}$, *the columns of* $\Pi^{(2)}_{D,h}(\cdot, \mathbf{y})$ *and the rows of* $\Pi^{(1)}_{D,h}(\mathbf{x}, \cdot)$ *are radiating solutions to the Navier equation in* U_H.

Proof: A direct consequence of the previous theorem by application of Lemma 2.5. ■

Much attention has also been paid to the special case of scattering by a periodic surface, i.e. a diffraction grating. In this case, one usually imposes a radiation condition using the Rayleigh expansion [5, 27, 36, 37].

Definition 4.7 *Assume f to be 2π-periodic. Then $\mathbf{u} \in BC(\Omega)$ is said to satisfy the* Rayleigh expansion radiation condition (RERC) *if, for $x_2 > \max f$ it has an expansion of the form*

$$\mathbf{u}(\mathbf{x}) = \sum_{n\in\mathbb{Z}} \left\{ u_{p,n} \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} \mathrm{e}^{i(\alpha_n x_1 + \beta_n x_2)} + u_{s,n} \begin{pmatrix} \gamma_n \\ -\alpha_n \end{pmatrix} \mathrm{e}^{i(\alpha_n x_1 + \gamma_n x_2)} \right\},$$

where $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $u_{p,n}$, $u_{s,n} \in \mathbb{C}$ $(n \in \mathbb{Z})$, $\alpha_n := \alpha + n$,

$$\beta_n := \left\{ \begin{array}{ll} \sqrt{k_p^2 - \alpha_n^2}, & \alpha_n^2 \leq k_p^2 \\ i\sqrt{\alpha_n^2 - k_p^2}, & \alpha_n^2 > k_p^2 \end{array} \right., \qquad \gamma_n := \left\{ \begin{array}{ll} \sqrt{k_s^2 - \alpha_n^2}, & \alpha_n^2 \leq k_s^2 \\ i\sqrt{\alpha_n^2 - k_s^2}, & \alpha_n^2 > k_s^2 \end{array} \right. .$$

A field $\mathbf{u}$ satisfying the RERC is quasi-periodic with phase-shift α *in $U_{\max f}$ and thus also, by analytic continuation, in Ω, that is, for all $\mathbf{x} = (x_1, x_2)^\top \in \Omega$,*

$$\mathbf{u}(x_1 + 2\pi, x_2) = \mathrm{e}^{i\alpha 2\pi}\, \mathbf{u}(x_1, x_2).$$

As a periodic surface is a special case of a rough surface, the problem of scattering by a diffraction grating can be seen as a special case of the problem of scattering by a rough surface. The radiation condition that will be introduced subsequently for scattering problems involving a rough surface will thus have to be satisfied by vector fields satisfing the RERC as well as by radiating fields in the sense of Definition 4.2. It will be shown that this is indeed the case.

4.2 A New Radiation Condition for Scattering by Rough Surfaces

We will start this section with a brief review of some results for the scalar case. In this case, the governing equation is the Helmholtz equation,

$$\Delta\, u + k^2\, u = 0.$$

Its free field Green's function is

$$\Phi(\mathbf{x},\mathbf{y}) := \frac{i}{4}\, H_0^{(1)}(k\,|\mathbf{x}-\mathbf{y}|).$$

There has been considerable progress on such scalar wave scattering problems involving rough surfaces and inhomogeneous layers [11,15,19–21,48]. The foundation of much of these results is a new radiation condition, termed the *upward propagating radiation condition:*

Definition 4.8 *A solution $u : G \to \mathbb{C}$ to the Helmholtz equation in $G \subset \mathbb{R}^2$ is said to satisfy* the upwards propagating radiation condition (UPRC)*, if, for some $H \in \mathbb{R}$ and $\phi \in L^\infty(T_H)$, $U_H \subset G$ and*

$$u(\mathbf{x}) = 2\int_{T_H} \frac{\partial \Phi}{\partial y_2}(\mathbf{x},\mathbf{y})\,\phi(\mathbf{y})\,ds(\mathbf{y}), \qquad \mathbf{x} \in U_H.$$

For the elastic wave case, a similar approach will be adopted. For the definition of a UPRC, we will employ the matrix of fundamental solutions $\Gamma_{D,h}$:

Definition 4.9 *A solution $\mathbf{u} : G \to \mathbb{C}^2$ to the Navier equation (2.4) in $G \subset \mathbb{R}^2$ is said to satisfy the* upward propagating radiation condition (UPRC)*, if, for some $H \in \mathbb{R}$ and $\phi \in [L^\infty(T_H)]^2$, $U_H \subset G$ and*

$$\mathbf{u}(\mathbf{x}) = \int_{T_H} \Pi^{(2)}_{D,H}(\mathbf{x},\mathbf{y})\,\phi(\mathbf{y})\,ds(\mathbf{y}), \qquad \mathbf{x} \in U_H. \tag{4.2}$$

Remark 4.10 *Note that from Theorem 2.16 (a) it follows that for arbitrary $\phi \in [L^\infty(T_h)]^2$ the integral in (4.2) exists as an improper integral.*

Remark 4.11 *Apparently the definition of the upward propagating radiation condition depends on the choice of the parameters $\tilde{\lambda}$ and $\tilde{\mu}$ in the definition of the generalised stresses. However, Theorem 4.12 below shows that the definition and the density ϕ itself are in fact independent of these numbers.*

The following theorem characterises the UPRC further and also establishes, through the equivalence of (a) and (c), that it is satisfied by any radiating solution (c.f. the charactarisation of the scalar UPRC in [19, Theorem 2.9]):

Theorem 4.12 *Given $a \in \mathbb{R}$ and $\mathbf{u} : U_a \to \mathbb{C}^2$, the following statements are equivalent:*

(a) $\mathbf{u} \in [C^2(U_a)]^2$, $\mathbf{u} \in [L^\infty(U_a \setminus U_H)]^2$ *for all* $H > a$, $\Delta^* \mathbf{u} + \omega^2 \mathbf{u} = 0$ *in* U_a, *and* $\mathbf{u}$ *satisfies the UPRC in* U_a.

(b) $\mathbf{u} \in [C^2(U_a)]^2$, $\mathbf{u} \in [L^\infty(U_a \setminus U_H)]^2$ *for all* $H > a$, $\Delta^* \mathbf{u} + \omega^2 \mathbf{u} = 0$ *in* U_a, *and for some* $H > a$ *and* $\phi_1,\ \phi_2 \in L^\infty(T_H)$,

$$\mathbf{u}(\mathbf{x}) = 2\,\mathrm{grad} \int\limits_{T_H} \frac{\partial \Phi_p}{\partial y_2}(\mathbf{x},\mathbf{y})\,\phi_1(\mathbf{y})\,ds(\mathbf{y}) + 2\,\mathrm{grad}^\perp \int\limits_{T_H} \frac{\partial \Phi_s}{\partial y_2}(\mathbf{x},\mathbf{y})\,\phi_2(\mathbf{y})\,ds(\mathbf{y})$$

for all $\mathbf{x} \in U_H$ *where* Φ_p *and* Φ_s *denote the fundamental solutions for the Helmholtz equation with* k *replaced by* k_p *and* k_s, *respectively.*

(c) $\mathbf{u} \in [L^\infty(U_a \setminus U_H)]^2$ *for all* $H > a$ *and there exists a sequence* $(\mathbf{u}_n)$ *of radiating solutions such that* $\mathbf{u}_n(\mathbf{x}) \to \mathbf{u}(\mathbf{x})$ *uniformly on compact subsets of* U_a *and*

$$\sup_{\mathbf{x}\in U_H \setminus U_{h'},\, n \in N} |\mathbf{u}_n(\mathbf{x})| < \infty \tag{4.3}$$

for all $H, h' \in \mathbb{R}$ *satisfying* $h' > H > a$.

(d) $\mathbf{u}$ *satisfies (4.2) for* $H = a$ *and some* $\phi \in [L^\infty(T_a)]^2$.

(e) $\mathbf{u} \in [L^\infty(U_a \setminus U_H)]^2$ *for some* $H > a$ *and* $\mathbf{u}$ *satisfies (4.2) for each* $H > a$ *with* $\phi = \mathbf{u}|_{T_H}$.

(f) $\mathbf{u} \in [C^2(U_a)]^2$, $\mathbf{u} \in [L^\infty(U_a \setminus U_H)]^2$ *for all* $H > a$, $\Delta^* \mathbf{u} + \omega^2 \mathbf{u} = 0$ *in* U_a, *and for every* $H > a$ *and radiating solution in* U_a, $\mathbf{w}$, *such that the restrictions of* $\mathbf{w}$ *and* $\mathbf{Pw}$ *to* T_H *are in* $[L^1(T_H)]^2$, *there holds*

$$\int_{T_H} (\mathbf{u} \cdot \mathbf{Pw} - \mathbf{w} \cdot \mathbf{Pu})\,ds = 0. \tag{4.4}$$

Proof: (a) $\Rightarrow$ (b): With H chosen so that (4.2) holds, we introduce the functions

$$\begin{array}{rcl} \Psi_{p,k}(\mathbf{x},\mathbf{y}) & := & -\frac{1}{k_p^2}\,\mathrm{div}_{\mathbf{x}}\,\Pi^{(2)}_{D,H,\cdot k}(\mathbf{x},\mathbf{y}) \\ \Psi_{s,k}(\mathbf{x},\mathbf{y}) & := & -\frac{1}{k_s^2}\,\mathrm{div}^\perp_{\mathbf{x}}\,\Pi^{(2)}_{D,H,\cdot k}(\mathbf{x},\mathbf{y}) \end{array} \qquad k = 1, 2,$$

and rewrite $\mathbf{u}(\mathbf{x})$ for $\mathbf{x} \in U_H$ as

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \mathbf{u}_p(\mathbf{x}) + \mathbf{u}_s(\mathbf{x}) \\ &= \mathrm{grad} \int_{T_H} \sum_{k=1}^{2} \Psi_{p,k}(\mathbf{x},\mathbf{y})\,\phi_k(\mathbf{y})\,ds(\mathbf{y}) \\ &\quad + \mathrm{grad}^\perp \int_{T_H} \sum_{k=1}^{2} \Psi_{s,k}(\mathbf{x},\mathbf{y})\,\phi_k(\mathbf{y})\,ds(\mathbf{y}). \end{aligned} \tag{4.5}$$

Limiting our attention to the first integral for the moment, we define

$$\begin{aligned}\mathbf{v}_N(\mathbf{x}) &= \int_{T_H(N)} \sum_{k=1}^{2} \Psi_{p,k}(\mathbf{x},\mathbf{y})\,\phi_k(\mathbf{y})\,ds(\mathbf{y}),\\ \mathbf{v}(\mathbf{x}) &= \int_{T_H} \sum_{k=1}^{2} \Psi_{p,k}(\mathbf{x},\mathbf{y})\,\phi_k(\mathbf{y})\,ds(\mathbf{y}).\end{aligned}$$

For $H' > H$, the vector fields $\mathbf{v}_N$ are solutions to the Helmholtz equation $\Delta\,\mathbf{v}_N + k_p^2\mathbf{v}_N = 0$ in $U_{H'}$. By Corollary 4.6 and one applications of Lemma 2.5 we see that they are furthermore radiating in $U_{H'}$. By Theorem 2.13 together with Lemma 2.5, there also holds $\mathbf{v}_N(\mathbf{x}) \to \mathbf{v}(\mathbf{x})$ uniformly on compact subsets of $U_{H'}$. For $h' > H'$, by Remark 2.17, we finally see that

$$\sup_{\mathbf{x}\in U_{H'}\setminus U_{h'},\,n\in\mathbb{N}} |\mathbf{v}_N(\mathbf{x})| < \infty.$$

So by Theorem 2.1 in [19], $\mathbf{v}$ satisfies the UPRC for the Helmholtz equation (see Definition 4.8), which is the assertion. The argument for the second integral in (4.5) is identical.

(b) $\Rightarrow$ (c): Set $\Psi_1 := -1/k_p^2 \operatorname{div}\mathbf{u}$ and $\Psi_2 := -1/k_s^2 \operatorname{div}^\perp \mathbf{u}$. Then (b) implies, that for all $\mathbf{x} \in U_H$ there holds

$$\begin{aligned}\Psi_1(\mathbf{x}) &= 2\int_{T_H} \frac{\partial \Phi_p}{\partial y_2}(\mathbf{x},\mathbf{y})\,\phi_1(\mathbf{y})\,ds(\mathbf{y}),\\ \Psi_2(\mathbf{x}) &= 2\int_{T_H} \frac{\partial \Phi_s}{\partial y_2}(\mathbf{x},\mathbf{y})\,\phi_2(\mathbf{y})\,ds(\mathbf{y}).\end{aligned}$$

From the equivalence of (i) and (ii) in Theorem 2.9 in [19], it follows that there exist sequences $(\Psi_j^{(n)})$ $(j = 1, 2)$ of radiating solutions to the Helmholtz equation with $k = k_p$ and $k = k_s$ respectively such that $\Psi_j^{(n)}(\mathbf{x}) \to \Psi_j(\mathbf{x})$ uniformly on compact subsets of U_a and

$$\sup_{\mathbf{x}\in U_a\setminus U_h,\,n\in N,\,j=1,2} |\Psi_j^{(n)}(\mathbf{x})| < \infty$$

for all $h > a$. Set

$$\mathbf{u}_n(\mathbf{x}) := \operatorname{grad}\Psi_1^{(n)}(\mathbf{x}) + \operatorname{grad}^\perp \Psi_2^{(n)}(\mathbf{x}).$$

Lemma 2.5 then implies (4.3) and that $\mathbf{u}_n(\mathbf{x})$ converges to $\mathbf{u}(\mathbf{x})$ uniformly on compact subsets of U_a.

(c) $\Rightarrow$ (f): Suppose $H > a$ and set $D := U_H \cap B_R(0)$ for some $R > H$, where $B_R(0)$ denotes the open ball with centre 0 and radius R. Further assume $\mathbf{w}$ to be a radiating solution in U_a, such that the restrictions of $\mathbf{w}$ and $\mathbf{P}\mathbf{w}$ to T_H are in $[L^1(T_H)]^2$. Then

$$\int_{\partial D} \mathbf{u}_n \cdot \mathbf{P}\mathbf{w} - \mathbf{w}\cdot\mathbf{P}\mathbf{u}_n\,ds = 0$$

follows by the third generalised Betti formula (2.10). Letting $R \to \infty$ and using the fact that $\mathbf{w}$ and $\mathbf{u}_n$ are radiating solutions to the Navier equation, we conclude

$$\int_{T_H} \mathbf{u}_n \cdot \mathbf{P}\mathbf{w} - \mathbf{w} \cdot \mathbf{P}\mathbf{u}_n \, ds = 0.$$

Taking the limit as $n \to \infty$, recalling (4.3) and using Lemma 2.5, we see that (4.4) holds. The remaining assertion follows from Corollary 2.6.

(f) $\Rightarrow$ (a),(e): It suffices to show that (4.2) holds for all $H > a$ with $\phi = \mathbf{u}|_{T_H}$.

Given $H > a$ and $\mathbf{x} \in U_H$, choose $h', A \in \mathbb{R}$ with $h' > x_2 > H$ and $A > |x_1|$. Set $B := \{\mathbf{y} \in U_H \setminus \overline{U}_{h'} : |y_1| < A\}$. Then, by Theorem 2.16 (e),

$$\mathbf{u}(\mathbf{x}) = \int_{\partial B} \left\{ \Gamma_{D,H}(\mathbf{x},\mathbf{y})\, \mathbf{P}\mathbf{u}(\mathbf{y}) - \Pi^{(2)}_{D,H}(\mathbf{x},\mathbf{y})\, \mathbf{u}(\mathbf{y}) \right\} ds(\mathbf{y}).$$

Letting $A \to \infty$ and recalling $\mathbf{u} \in [L^\infty(U_a \setminus U_{h'})]^2$ as well as Theorems 2.10 and 4.5 as well as Theorem 2.16 (a), we obtain that

$$\begin{aligned} \mathbf{u}(\mathbf{x}) \;=\; & \int_{T_{h'}} \left\{ \Gamma_{D,H}(\mathbf{x},\mathbf{y})\, \mathbf{P}\mathbf{u}(\mathbf{y}) - \Pi^{(2)}_{D,H}(\mathbf{x},\mathbf{y})\, \mathbf{u}(\mathbf{y}) \right\} ds(\mathbf{y}) \\ & + \int_{T_H} \Pi^{(2)}_{D,H}(\mathbf{x},\mathbf{y})\, \mathbf{u}(\mathbf{y})\, ds(\mathbf{y}). \end{aligned}$$

By applying (4.4) with $\mathbf{w}$ equal to each of the rows of $\Gamma_{D,H}(\mathbf{x},\cdot)$ in turn, the integral over $T_{h'}$ is seen to vanish.

(e) $\Rightarrow$ (d): Introducing, for $\alpha \in \mathbb{R}$, the mapping

$$\eta_\alpha(\mathbf{z}) := (z_1, z_2 + \alpha)^\top,$$

we have from (e) that

$$\mathbf{u}(\mathbf{x}) = \int_{T_a} \Pi^{(2)}_{D,H}(\mathbf{x}, \eta_{H-a}(\mathbf{z}))\, \mathbf{u}(\eta_{H-a}(\mathbf{z}))\, ds(\mathbf{z}), \qquad \mathbf{x} \in U_H. \tag{4.6}$$

As $\mathbf{u} \in [L^\infty(U_a \setminus U_H) \cap C(U_a)]^2$ for some $H > a$, the densities $\mathbf{u}(\eta_{H-a}(\cdot))$ are all in some ball in $[L^\infty(T_a)]^2$ for H close enough to a. Recalling that the unit ball in $[L^\infty(T_a)]^2$ is weak$*$ sequentially compact, there thus exists a sequence (H_n) with $H_n \to a$ and $\mathbf{u}(\eta_{H_n-a}(\cdot)) \to \phi \in [L^\infty(T_a)]^2$. Taking the limit as $H \to a$, through this sequence in (4.6) we now conclude that (4.2) holds for $H = a$ with this ϕ.

(d) $\Rightarrow$ (c): As (4.2) is satisfied with $h = a$, it follows from Theorem 2.16 (a) that

$$|\mathbf{u}(\mathbf{x})| \le \|\phi\|_\infty\, g(x_2), \qquad \mathbf{x} \in U_a, \tag{4.7}$$

where $g \in C(\mathbb{R})$. Setting

$$\mathbf{u}_n(\mathbf{x}) := \int_{T_a(n)} \Pi^{(2)}_{D,a}(\mathbf{x},\mathbf{y})\,\phi(\mathbf{y})\,ds(\mathbf{y}), \qquad \mathbf{x} \in U_a,$$

$\mathbf{u} \in [L^\infty(U_a \setminus U_h)]^2$ for all $h > a$ and (4.3) follow from (4.7). That $\mathbf{u}_n(\mathbf{x})$ converges to $\mathbf{u}(\mathbf{x})$ uniformly on compact subsets of U_a and that $\mathbf{u}_n$ is radiating, is also easily seen from Theorem 2.16 (a). ∎

Remark 4.13 *From Remark 2.14 in [19] and the equivalence of statements (a) and (b) in Theorem 4.12, it follows that any bounded solution to the Navier equation* $\mathbf{u}$ *in* Ω *that satisfies the UPRC in* Ω *and is quasi-periodic in* Ω *also satisfies the RERC of Definition 4.7. Conversely, applying the same results, a bounded, quasi-periodic solution to the Navier equation in* Ω*, satisfying the RERC, also satisfies the UPRC.*

It remains to be shown that the elastic single- and double-layer potentials $\mathbf{v}$ defined by (3.1) and $\mathbf{w}$ defined by (3.2) satisfy the UPRC. To this end, assume $f \in C^{1,1}(\mathbb{R})$, $\phi \in [BC(S)]^2$ and introduce the vector fields $\mathbf{v}_N$ and $\mathbf{w}_N$, $N \in \mathbb{N}$, by

$$\left.\begin{array}{rcl} \mathbf{v}_N(\mathbf{x}) & := & \displaystyle\int_{S(N)} \Gamma_{D,h}(\mathbf{x},\mathbf{y})\,\phi(\mathbf{y})\,ds(\mathbf{y}), \\ \mathbf{w}_N(\mathbf{x}) & := & \displaystyle\int_{S(N)} \Pi^{(2)}_{D,h}(\mathbf{x},\mathbf{y})\,\phi(\mathbf{y})\,ds(\mathbf{y}), \end{array}\right. \qquad \mathbf{x} \in \Omega.$$

As S_N is a bounded set, applications of Theorem 4.5 and Corollary 4.6 now yield that v_N and w_N are radiating solutions to the Navier equation in the sense of Definition 4.2 for any $H > \sup f$. From Theorems 2.13 and 2.16 (a), the sequences $(\mathbf{v}_N(\mathbf{x}))$ and $(\mathbf{w}_N(\mathbf{x}))$ converge to $\mathbf{v}(\mathbf{x})$ and $\mathbf{w}(\mathbf{x})$, $\mathbf{x} \in \Omega$, respectively, and this convergence is uniform on compact subsets of U_a for any $a > \sup f$. We can finally apply Theorem 3.11 (a) and Theorem 3.12 (a) to see that (4.3) holds with $a > \sup f$. Thus, the equivalence of (a) and (c) in Theorem 4.12 yields the following result:

Theorem 4.14 *Assume* $f \in C^{1,1}(\mathbb{R})$*. Then, the elastic single-layer potential* $\mathbf{v}$ *and the elastic double-layer potential* $\mathbf{w}$ *satisfy the UPRC.*

4.3 Formulation as a Boundary Value Problem and Uniqueness of Solution

Recall the formulation of the rough surface scattering problem from Section 1.1:

> *Scattering Problem:* Given an incident field $\mathbf{u}^{inc}$ that is a solution to the Navier equation (2.4) in Ω, find the scattered field $\mathbf{u}$ such that $\mathbf{u}^{inc}+\mathbf{u} = 0$ on S.

We will now formulate this scattering problem as a boundary value problem. In fact, the assumptions can be formulated slightly more generally: we only require that $\mathbf{u}^{inc}$ is a solution to the Navier equation in some neighborhood of $S = \partial\Omega$ and that $\mathbf{g} := -\mathbf{u}^{inc}|_S \in [BC(S) \cap H^{1/2}_{loc}(S)]^2$.

Problem 4.15 *Find a vector field* $\mathbf{u} \in [C^2(\Omega) \cap C(\overline{\Omega}) \cap H^1_{loc}(\Omega)]^2$ *that satisfies*

1. *the Navier equation* $\Delta^* \mathbf{u} + \omega^2 \mathbf{u} = 0$ *in* Ω*,*
2. *the Dirichlet boundary condition* $\mathbf{u} = \mathbf{g}$ *on* S *for some vector field* $\mathbf{g} \in [BC(S) \cap H^{1/2}_{loc}(S)]^2$*,*
3. *the vertical growth rate condition*
$$\sup_{\mathbf{x}\in\Omega} x_2^{\beta} |\mathbf{u}(\mathbf{x})| < \infty \tag{4.8}$$
for some $\beta \in \mathbb{R}$ *and*
4. *the UPRC in* Ω*.*

Remark 4.16 *A solution of Problem 4.15 satisfies statement (a) of Theorem 4.12 with any* $a > \sup f$*.*

The remaining part of this chapter will be devoted to proving that Problem 4.15 has at most one solution. The question of existence of solution will be postponed until Chapter 5. However, it is worth pointing out at this stage, that it is clear *a priori* that Problem 4.15 is well posed in certain, non-trivial cases: From Remark 4.13 we see that, in the case when f is periodic and the Dirichlet data $\mathbf{g}$ is quasi-periodic, Problem 4.15 reduces to the diffraction grating problem considered in [4, 5], if we assume additionally that the solution $\mathbf{u}$ is quasi-periodic. The diffraction grating problem was shown in [4, 5] to be uniquely solvable. Thus we know that Problem 4.15 admits solutions, in the case when f is periodic and $\mathbf{g}$ quasi-periodic.

Let us also point out that, as we will now proceed to show that Problem 4.15 has at most one solution in every case, we will in fact prove for the diffraction grating problem that the additional requirement that the solution be quasi-periodic can be dropped.

In all of what follows, let h denote a real number with $h < \inf f$. The first step in the uniqueness proof will be the following representation theorem:

Theorem 4.17 *Let* $\mathbf{u}$ *be a solution to Problem 4.15 with* $\mathbf{g} \equiv 0$. *Then* $\mathbf{u} \in [C^1(\bar{\Omega})]^2$, *its first derivatives are bounded in* D_H *for any* $H > \sup f$ *and*

$$\mathbf{u}(\mathbf{x}) = -\int_S \Gamma_{D,h}(\mathbf{x},\mathbf{y})\, \mathbf{P}\mathbf{u}(\mathbf{y})\, ds(\mathbf{y})$$

holds for all $\mathbf{x} \in \Omega$.

Proof: That $\mathbf{u} \in [C^1(\bar{\Omega})]^2$ and its first derivatives are bounded in D_H, $H > \sup f$, is a consequence of Theorem 2.7. For $\mathbf{x} \in \Omega$, choose $h', A \in \mathbb{R}$ and $\varepsilon > 0$ with $h' > \max\{x_2, \sup f\}$ and $A > |x_1|$. Define $D^A_{\varepsilon,h'} := \{\mathbf{x} \in D_{h'}(A) : x_2 > f(x_1) + \varepsilon\}$. Then, by Theorem 2.16 (e), there holds

$$\mathbf{u}(\mathbf{x}) = \int_{\partial D^A_{\varepsilon,h'}} \left\{ \Gamma_{D,h}(\mathbf{x},\mathbf{y})\, \mathbf{P}\mathbf{u}(\mathbf{y}) - \Pi^{(2)}_{D,h}(\mathbf{x},\mathbf{y})\, \mathbf{u}(\mathbf{y}) \right\} ds(\mathbf{y}).$$

By applying Theorem 2.7, we see that the growth rate condition (4.8) for $\mathbf{u}$ also holds for any first derivative of $\mathbf{u}$. Letting $A \to \infty$ and recalling Theorems 2.13 and 2.16 (a) then yields

$$\begin{aligned}\mathbf{u}(\mathbf{x}) \;&=\; \int_{T_{h'}} \left\{ \Gamma_{D,h}(\mathbf{x},\mathbf{y})\, \mathbf{P}\mathbf{u}(\mathbf{y}) - \Pi^{(2)}_{D,h}(\mathbf{x},\mathbf{y})\, \mathbf{u}(\mathbf{y}) \right\} ds(\mathbf{y}) \\ &\qquad - \int_{S_\varepsilon} \left\{ \Gamma_{D,h}(\mathbf{x},\mathbf{y})\, \mathbf{P}\mathbf{u}(\mathbf{y}) - \Pi^{(2)}_{D,h}(\mathbf{x},\mathbf{y})\, \mathbf{u}(\mathbf{y}) \right\} ds(\mathbf{y}),\end{aligned}$$

where $S_\varepsilon := \{\mathbf{x} \in \Omega : x_2 = f(x_1) + \varepsilon\}$. Note here, that the normal $\mathbf{n}$ to S_ε is assumed to be pointing upwards. The proof is now completed by recalling Remark 4.16 and the equivalence of (a) and (f) in Theorem 4.12, by which the integral over $T_{h'}$ vanishes, and letting $\varepsilon \to 0$. ■

Let us now introduce some functionals that will be of importance in the following arguments. Let $h' > \sup f$, $A > 0$ and $\mathbf{u} \in C^1(\bar{\Omega})$. We define

$$\begin{aligned}I(h',A)[\mathbf{u}] \;&:=\; \int_{T_{h'}(A)} (2\mu+\lambda)\left(\left|\frac{\partial u_2}{\partial x_2}\right|^2 - \left|\frac{\partial u_1}{\partial x_1}\right|^2 \right) \\ &\qquad\qquad + \mu \left(\left|\frac{\partial u_1}{\partial x_2}\right|^2 - \left|\frac{\partial u_2}{\partial x_1}\right|^2 \right) + \omega^2 |\mathbf{u}|^2\, ds, \\ J_1(h',A)[\mathbf{u}] \;&:=\; 2\,\mathrm{Re} \int_{\gamma(h',A)} \frac{\partial \bar{\mathbf{u}}}{\partial x_2} \cdot \mathbf{P}\mathbf{u}\, ds, \\ J_2(h',A)[\mathbf{u}] \;&:=\; \mathrm{Im} \int_{\gamma(h',A)} \bar{\mathbf{u}} \cdot \mathbf{P}\mathbf{u}\, ds, \\ K(h',A)[\mathbf{u}] \;&:=\; \mathrm{Im} \int_{T_{h'}(A)} \bar{\mathbf{u}} \cdot \mathbf{P}\mathbf{u}\, ds,\end{aligned}$$

where we recall the assumptions on the direction of the normal vectors in Section 1.3. We will now investigate the properties of these functionals.

Lemma 4.18 *Suppose* $\mathbf{u}$ *satisfies statement (b) in Theorem 4.12 with* $H > \sup f$ *and some densities* $\phi_j \in L^2(T_H) \cap L^\infty(T_H)$ $(j = 1, 2)$. *Then, for all* $h' > H$, *there holds*

$$I(h', \infty)[\mathbf{u}] = 2\omega^2 \left\{ \int_{-k_p}^{k_p} |\tilde{\phi}_1|^2 \gamma_p^2 \, dt + \int_{-k_s}^{k_s} |\tilde{\phi}_2|^2 \gamma_s^2 \, dt \right\},$$

where $\tilde{\phi}_1(t) := \mathrm{e}^{-i\gamma_p H} \hat{\phi}_1(t)$, $\tilde{\phi}_2(t) := \mathrm{e}^{-i\gamma_s H} \hat{\phi}_2(t)$ *and* $\hat{\phi}_j$ *denotes the Fourier transform of* $\phi_j(y_1, y_2)$ *with respect to* y_1 $(j = 1, 2)$, *i.e.*

$$\hat{\phi}_j(t) = \int_{-\infty}^{\infty} \phi_j(y_1, y_2) \, \mathrm{e}^{iy_1 t} \, dy_1.$$

Proof: Choose $H > \sup f$ so that the representation for $\mathbf{u}$ in U_H according to statement (b) of Theorem 4.12 holds. Then the argument presented for the derivation of equation (29) in [11] yields

$$\mathbf{u}(\mathbf{x}) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \tilde{\phi}_1(t) \begin{pmatrix} t \\ \gamma_p \end{pmatrix} \mathrm{e}^{i(tx_1 + \gamma_p x_2)} + \tilde{\phi}_2(t) \begin{pmatrix} \gamma_s \\ -t \end{pmatrix} \mathrm{e}^{i(tx_1 + \gamma_s x_2)} \, dt, \qquad \mathbf{x} \in U_H.$$

By an application of Parseval's Theorem we derive from this representation, for any $h' > H$,

$$\begin{aligned}
\int_{T_{h'}} |u_1|^2 ds &= \int_{-\infty}^{\infty} |t\tilde{\phi}_1 + \gamma_s \tilde{\phi}_2|^2 dt, \\
\int_{T_{h'}} |u_2|^2 ds &= \int_{-\infty}^{\infty} |\gamma_p \tilde{\phi}_1 - t\tilde{\phi}_2|^2 dt, \\
\int_{T_{h'}} \left| \frac{\partial u_1}{\partial x_1} \right|^2 ds &= \int_{-\infty}^{\infty} |it^2 \tilde{\phi}_1 + it\gamma_s \tilde{\phi}_2|^2 dt, \\
\int_{T_{h'}} \left| \frac{\partial u_1}{\partial x_2} \right|^2 ds &= \int_{-\infty}^{\infty} |it\gamma_p \tilde{\phi}_1 + i\gamma_s^2 \tilde{\phi}_2|^2 dt, \\
\int_{T_{h'}} \left| \frac{\partial u_2}{\partial x_1} \right|^2 ds &= \int_{-\infty}^{\infty} |it\gamma_p \tilde{\phi}_1 - it^2 \tilde{\phi}_2|^2 dt, \\
\int_{T_{h'}} \left| \frac{\partial u_2}{\partial x_2} \right|^2 ds &= \int_{-\infty}^{\infty} |i\gamma_p^2 \tilde{\phi}_1 - it\gamma_s \tilde{\phi}_2|^2 dt.
\end{aligned}$$

From these formulae, the following three identities are easily obtained:

$$\begin{aligned}\int_{T_{h'}} \left\{|u_1|^2 + |u_2|^2\right\} ds &= \int_{-\infty}^{\infty} |\tilde{\phi}_1|^2 \,(t^2 + |\gamma_p|^2) + \tilde{\phi}_1\, \overline{\tilde{\phi}_2}\,(t\overline{\gamma_s} - t\gamma_p) \\ &\quad + \overline{\tilde{\phi}_1}\, \tilde{\phi}_2\,(t\gamma_s - t\overline{\gamma_p}) + |\tilde{\phi}_2|^2\,(|\gamma_s|^2 + t^2)\, dt, \quad (4.9)\end{aligned}$$

$$\begin{aligned}\int_{T_{h'}} \left\{\left|\frac{\partial u_2}{\partial x_2}\right|^2 + \left|\frac{\partial u_1}{\partial x_1}\right|^2\right\} ds &= \int_{-\infty}^{\infty} |\tilde{\phi}_1|^2\,(\gamma_p^4 - t^4) + \tilde{\phi}_1\,\overline{\tilde{\phi}_2}\,(-t\gamma_p^2\overline{\gamma_s} - t^3\overline{\gamma_s}) \\ &\quad + \overline{\tilde{\phi}_1}\,\tilde{\phi}_2\,(-t\gamma_p^2\gamma_s - t^3\gamma_s)\, dt, \quad (4.10)\end{aligned}$$

$$\begin{aligned}\int_{T_{h'}} \left\{\left|\frac{\partial u_1}{\partial x_2}\right|^2 + \left|\frac{\partial u_2}{\partial x_1}\right|^2\right\} ds &= \int_{-\infty}^{\infty} \tilde{\phi}_1\,\overline{\tilde{\phi}_2}\,(t\gamma_p\gamma_s^2 + t^3\overline{\gamma_p}) + \overline{\tilde{\phi}_1}\,\tilde{\phi}_2\,(t\overline{\gamma_p}\gamma_s^2 - t^3\overline{\gamma_p}) \\ &\quad + |\tilde{\phi}_2|^2\,(\gamma_s^4 - t^4)\, dt. \quad (4.11)\end{aligned}$$

Combining (4.9)–(4.11) and observing $k_p^2 - t^2 + |\gamma_p|^2 = 0$ for $|t| > k_p$ as well as $k_s^2 - t^2 + |\gamma_s|^2 = 0$ for $|t| > k_s$ now yields the assertion. ■

Lemma 4.19 *Suppose all the assumptions of Lemma 4.18 are satisfied. Then, for all $h' > H$, we have the identity*

$$K(h', \infty)[\mathbf{u}] = \omega^2 \left\{\int_{-k_p}^{k_p} |\tilde{\phi}_1|^2 \gamma_p\, dt + \int_{-k_s}^{k_s} |\tilde{\phi}_2|^2 \gamma_s\, dt\right\}.$$

Proof: Recalling the remarks at the beginning of the proof of Lemma 4.18 and adopting the same notation, we easily see that

$$\begin{aligned}\int_{T_{h'}} \bar{\mathbf{u}} \cdot \mathbf{P}\mathbf{u}\, ds &= \int_{-\infty}^{\infty} \Big\{|\tilde{\phi}_1|^2 (i\mu t^2\gamma_p + i\tilde{\mu} t^2\gamma_p + i\tilde{\lambda} t^2\overline{\gamma_p} + i(2\mu+\lambda)\overline{\gamma_p}\gamma_p^2) \\ &\quad + \tilde{\phi}_1\overline{\tilde{\phi}_2}(i\mu t\gamma_p\overline{\gamma_s} + i\tilde{\mu} t\gamma_p\overline{\gamma_s} - i\tilde{\lambda} t^3 - i(2\mu+\lambda)t\gamma_p^2) \\ &\quad + \overline{\tilde{\phi}_1}\tilde{\phi}_2(i\mu t\gamma_s^2 - i\tilde{\mu} t^3 + i\tilde{\lambda} t\overline{\gamma_p}\gamma_s - i(2\mu+\lambda)t\overline{\gamma_p}\gamma_s) \\ &\quad + |\tilde{\phi}_2|(i\mu\gamma_s^2\overline{\gamma_s} - i\tilde{\mu} t^2\overline{\gamma_s} - i\tilde{\lambda} t^2\gamma_s + i(2\mu+\lambda)t^2\gamma_s)\Big\}\, dt.\end{aligned} \quad (4.12)$$

Taking the imaginary part, straightforward calculations show that all the terms involving $\tilde{\phi}_1\overline{\tilde{\phi}_2}$ and its complex conjugate cancel. On the other hand,

$$\mathrm{Im} \int_{-\infty}^{\infty} |\tilde{\phi}_1|^2 (i\mu t^2\gamma_p + iat^2\gamma_p + ibt^2\overline{\gamma_p} + i(2\mu+\lambda)\overline{\gamma}_p\gamma_p^2)\, dt$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{-\infty}^{\infty} |\tilde{\phi}_1|^2 \,(\mu t^2\gamma_p + \mu t^2\overline{\gamma_p} + \tilde{\mu} t^2\gamma_p + \tilde{\mu} t^2\overline{\gamma_p} \\
&\qquad + \tilde{\lambda} t^2\overline{\gamma_p} + \tilde{\lambda} t^2\gamma_p + (2\mu+\lambda)\overline{\gamma_p}\gamma_p^2 + (2\mu+\lambda)\gamma_p^3 \, dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty} (2\mu+\lambda)\, |\tilde{\phi}_1|^2 \,(t^2\gamma_p + \gamma_p^3 + t^2\overline{\gamma_p} + \overline{\gamma_p}\gamma_p^2)\, dt \\
&= \int_{-k_p}^{k_p} (2\mu+\lambda) k_p^2 \gamma_p |\tilde{\phi}_1|^2 \, dt
\end{aligned}
$$

A similar calculation for the remaining term in the expression (4.12) now yields the lemma. ■

The following corollary to the previous two lemmas is of fundamental importance:

Corollary 4.20 *Suppose all the assumptions of Lemma 4.18 are satisfied. Then, for all $h' > H$, there holds*

$$I(h', \infty)[\mathbf{u}] \le 2k_s\, K(h', \infty)[\mathbf{u}].$$

Proof: The assertion follows from Lemmas 4.18 and 4.19 by noting $k_p \ge \gamma_p$ on $[-k_p, k_p]$, $k_s \ge \gamma_s$ on $[-k_s, k_s]$ and $k_s > k_p$. ■

Another, simpler relation involving these functionals which is much easier to prove is stated in the following lemma:

Lemma 4.21 *Let $\mathbf{u}$ be a solution to Problem 4.15 with $\mathbf{g} \equiv 0$. Further assume $h' > \max f$ and $A > 0$. Then*

$$K(h', A)[\mathbf{u}] = -J_2(h', A)[\mathbf{u}].$$

Proof: Apply the third generalised Betti formula (2.10) to $\mathbf{u}$ and $\bar{\mathbf{u}}$ in $D_{h'}(A)$. ■

Assume now that $\mathbf{u}$ is a solution to Problem 4.15 with $\mathbf{g} \equiv 0$. As $\mathbf{u}$ and its tangential derivatives vanish on S, $\mathbf{Pu}$ has the simple form

$$\mathbf{Pu} = \mu \frac{\partial \mathbf{u}}{\partial \mathbf{n}} + (\lambda+\mu)\, \mathbf{n} \operatorname{div} \mathbf{u} \qquad \text{on } S, \tag{4.13}$$

and thus

$$\int_{S(A)} \frac{\partial \bar{\mathbf{u}}}{\partial x_2} \cdot \mathbf{Pu}\, ds = \int_{S(A)} \left\{ \mu\, n_2 \left| \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \right|^2 + (\lambda+\mu)\, n_2 \,|\operatorname{div} \mathbf{u}|^2 \right\} ds \tag{4.14}$$

for any $A > 0$. In a similar fashion, we also show that

$$\mathcal{E}_{\tilde{\mu},\tilde{\lambda}}(\bar{\mathbf{u}},\mathbf{u}) = \mu \left|\frac{\partial \mathbf{u}}{\partial \mathbf{n}}\right|^2 + (\lambda+\mu)\, |\mathrm{div}\,\mathbf{u}|^2 \qquad \text{on } S. \tag{4.15}$$

Combining (4.14) and (4.15), we conclude by an integration by parts that

$$\begin{aligned} 0 &\le \int_{S(A)} \left\{ \mu n_2 \left|\frac{\partial \mathbf{u}}{\partial \mathbf{n}}\right|^2 + (\lambda+\mu) n_2 |\mathrm{div}\,\mathbf{u}|^2 \right\} ds \\ &= 2\,\mathrm{Re} \int_{S(A)} \frac{\partial \bar{\mathbf{u}}}{\partial x_2} \cdot \mathbf{P}\mathbf{u}\, ds - \int_{S(A)} n_2\, \mathcal{E}_{\tilde{\mu},\tilde{\lambda}}(\bar{\mathbf{u}},\mathbf{u}) - n_2\,\omega^2 |\mathbf{u}|^2\, ds \\ &= 2\,\mathrm{Re} \int_{S(A)} \frac{\partial \bar{\mathbf{u}}}{\partial x_2} \cdot \mathbf{P}\mathbf{u}\, ds + 2\,\mathrm{Re} \int_{D_{h'}(A)} \mathcal{E}_{\tilde{\mu},\tilde{\lambda}}(\frac{\partial \bar{\mathbf{u}}}{\partial x_2},\mathbf{u}) - \omega^2 \frac{\partial \bar{\mathbf{u}}}{\partial x_2} \cdot \mathbf{u}\, d\mathbf{x} \\ &\qquad - \int_{T_{h'}(A)} \mathcal{E}_{\tilde{\mu},\tilde{\lambda}}(\bar{\mathbf{u}},\mathbf{u}) - \omega^2 |\mathbf{u}|^2\, ds \end{aligned} \tag{4.16}$$

On the other hand, for any $h' > \sup f$, by the first Betti formula (2.8) there also holds

$$\int_{\partial D_{h'}(A)} \frac{\partial \bar{\mathbf{u}}}{\partial x_2} \cdot \mathbf{P}\mathbf{u}\, ds = \int_{D_{h'}(A)} \mathcal{E}_{\tilde{\mu},\tilde{\lambda}}(\frac{\partial \bar{\mathbf{u}}}{\partial x_2},\mathbf{u}) - \omega^2 \frac{\partial \bar{\mathbf{u}}}{\partial x_2} \cdot \mathbf{u}\, d\mathbf{x}. \tag{4.17}$$

It is also not difficult to see that

$$\mathrm{Re} \int_{T_{h'}(A)} \left\{ 2\frac{\partial \bar{\mathbf{u}}}{\partial x_2} \cdot \mathbf{P}\mathbf{u} + \omega^2 |\mathbf{u}|^2 - \mathcal{E}_{\tilde{\mu},\tilde{\lambda}}(\bar{\mathbf{u}},\mathbf{u}) \right\} ds = I(h',A)[\mathbf{u}],$$

Combining this identity with (4.16) and (4.17), we conclude finally that

$$0 \le \int_{S(A)} \left\{ \mu n_2 \left|\frac{\partial \mathbf{u}}{\partial \mathbf{n}}\right|^2 + (\lambda+\mu) n_2 |\mathrm{div}\,\mathbf{u}|^2 \right\} ds = I(h',A)[\mathbf{u}] + J_1(h',A)[\mathbf{u}]. \tag{4.18}$$

The rest of the derivation of the uniqueness result is now a rather straightforward adaption of the method presented in [20] for the Helmholtz equation case. Let us introduce the vector fields $\mathbf{v}_A$ defined for $A > 0$ by

$$\mathbf{v}_A(\mathbf{x}) := - \int_{S(A)} \Gamma_{D,h}(\mathbf{x},\mathbf{y})\, \mathbf{P}\mathbf{u}(\mathbf{y})\, ds(\mathbf{y}).$$

Using the Cauchy-Schwarz inequality and Theorem 2.13, we find that $\mathbf{v}_A|_{h'} \in [L^2(T_{h'}) \cap BC(T_{h'})]^2$ for all $h' > \sup f$. As $\mathbf{v}_A$ is a radiating solution to the Navier equation for every $A \in \mathbb{R}$, it is seen to satisfy statement (c) of Theorem 4.12 and thus also statement (b) of that theorem. Thus, by Corollary 4.20,

$$I(h',\infty)[\mathbf{v}_A] \le 2k_s\, K(h',\infty)[\mathbf{v}_A]. \tag{4.19}$$

Now set $w(x_1) := |\mathbf{Pu}(x_1, f(x_1))|$ for $x_1 \in \mathbb{R}$. Then

$$\int_{-A}^{A} |w(x_1)|^2 \, dx_1 \le \int_{S(A)} |\mathbf{Pu}|^2 \, ds \le (1 + \|f'\|^2_{\infty;\mathbb{R}})^{1/2} \int_{-A}^{A} |w(x_1)|^2 \, dx_1 \tag{4.20}$$

follows. Using Theorem 2.13 and Lemma 2.5 we obtain the estimates

$$|\Gamma_{D,h}(\mathbf{x}, \mathbf{y})|, \ |\frac{\partial}{\partial x_j}\Gamma_{D,h}(\mathbf{x}, \mathbf{y})| \le C(1 + |x_1 - y_1|)^{-3/2} \qquad (j = 1, 2)$$

for $\mathbf{x} \in T_{h'}$, $\mathbf{y} \in S$, where C is some positive constant depending only on h' and h. This yields the estimates

$$\begin{array}{rcl} |\mathbf{v}_A(\mathbf{x})|, \ |\frac{\partial \mathbf{v}_A}{\partial x_j}(\mathbf{x})| & \le & C\, W_A(x_1) \\ |\mathbf{u}(\mathbf{x}) - \mathbf{v}_A(\mathbf{x})| & \le & C\,(W_\infty(x_1) - W_A(x_1)) \\ |\frac{\partial \mathbf{u}}{\partial x_j}(\mathbf{x}) - \frac{\partial \mathbf{v}_A}{\partial x_j}(\mathbf{x})| & \le & C\,(W_\infty(x_1) - W_A(x_1)) \end{array} \tag{4.21}$$

for $\mathbf{x} \in T_{h'}$, $j = 1, 2$, with certain generic constants C, where

$$W_A(x_1) := \int_{-A}^{A} (1 + |x_1 - y_1|)^{-3/2}\, w(y_1)\, dy_1, \qquad x_1 \in \mathbb{R}.$$

Recalling (4.13), we can estimate by (4.18)–(4.20) and Lemma 4.21

$$\begin{aligned} \int_{-A}^{A} |w(x_1)|^2 \, dx_1 \ &\le \ C \int_{S(A)} \left\{ \mu n_2 \left|\frac{\partial \mathbf{u}}{\partial \mathbf{n}}\right|^2 + (\lambda + \mu) n_2 |\mathrm{div}\, \mathbf{u}|^2 \right\} ds \\ &\le \ C \Big\{ |I(h', A)[\mathbf{u}] - I(h', A)[\mathbf{v}_A]| \\ &\qquad + |I(h', A)[\mathbf{v}_A] - I(h', \infty)[\mathbf{v}_A]| \\ &\qquad + 2k_s \Big[|K(h', \infty)[\mathbf{v}_A] - K(h', A)[\mathbf{v}_A]| \\ &\qquad\qquad + |K(h', A)[\mathbf{v}_A] - K(h', A)[\mathbf{u}]| \Big] \\ &\qquad + |J_1(h', A)[\mathbf{u}]| + 2k_s |J_2(h', A)[\mathbf{u}]| \Big\}. \end{aligned}$$

From (4.21), there now follows, with some positive constant C,

$$\left.\begin{array}{r} |I(h', A)[\mathbf{v}_A] - I(h', \infty)[\mathbf{v}_A|, \\ |K(h', \infty)[\mathbf{v}_A] - K(h', A)[\mathbf{v}_A]| \end{array}\right\} \le C \int_{\mathbb{R}\setminus[-A,A]} W_A^2(x_1)\, dx_1$$

and

$$\left.\begin{array}{r} |I(h', A)[\mathbf{u}] - I(h', A)[\mathbf{v}_A]|, \\ |K(h', A)[\mathbf{v}_A] - K(h', A)[\mathbf{u}]| \end{array}\right\} \le C \int_{-A}^{A} (W_\infty(x_1) - W_A(x_1))\, W_\infty(x_1)\, dx_1,$$

so that we finally conclude, for some constant $c > 0$ and all $A > 0$,

$$\begin{aligned}\int_{-A}^{A} |w(x_1)|^2\,dx_1 \;\leq\; c\Big\{ &\int_{\mathbb{R}\setminus[-A,A]} W_A^2(x_1)\,dx_1 \\ &+\int_{-A}^{A} (W_\infty(x_1)-W_A(x_1))\,W_\infty(x_1)\,dx_1 \\ &+|J_1(h',A)[\mathbf{u}]|+|J_2(h',A)[\mathbf{u}]|\Big\}. \qquad (4.22)\end{aligned}$$

Since (4.22) holds and we also have from Theorem 2.7 that $w \in L^\infty(\mathbb{R})$, we can apply Lemma A in [20] to obtain that $w \in L^2(\mathbb{R})$ and, noting (4.20), that for all $A_0 > 0$,

$$\begin{aligned}(1+\|f'\|^2_{\infty;\mathbb{R}})^{-1/2}\int_S |\mathbf{Pu}|^2\,ds \;&\leq\; \int_{-\infty}^{\infty} |w(x_1)|^2\,dx_1 \\ &\leq\; c\,\sup_{A>A_0}\{|J_1(h',A)[\mathbf{u}]|+|J_2(h',A)[\mathbf{u}]\}|. \qquad (4.23)\end{aligned}$$

For $x \in D_{h'}$ with $|x_1| > 0$, we now deduce by Theorem 2.13, Theorem 4.17 and the Cauchy-Schwarz inequality, that

$$\begin{aligned}|\mathbf{u}(\mathbf{x})|^2 \;&\leq\; 2\left\{\int_{S\setminus S(|x_1|/2)} |\Gamma_{D,h}(\mathbf{x},\mathbf{y})\,\mathbf{Pu}(\mathbf{y})|\,ds(\mathbf{y})\right\}^2 \\ &\quad + 2\left\{\int_{S(|x_1|/2)} |\Gamma_{D,h}(\mathbf{x},\mathbf{y})\,\mathbf{Pu}(\mathbf{y})|\,ds(\mathbf{y})\right\}^2 \\ &\leq\; C_1\int_{S\setminus S(|x_1|/2)} |\mathbf{Pu}|^2\,ds + C_2\left(\frac{|x_1|}{2}\right)^{-2},\end{aligned}$$

where

$$C_1 = 16\,\sup_{\mathbf{x}\in D_{h'}}\int_S \max_{j,k=1,2}|\Gamma_{D,h,jk}(\mathbf{x},\mathbf{y})|^2\,ds(\mathbf{y}) < \infty$$

by Remark 2.14 and

$$C_2 = 32\,\|\mathcal{H}\|^2_{C([0,h'-h]^2)}\,(1+\|f'\|_{\infty;\mathbb{R}})^{1/2}\,\|\mathbf{Pu}\|_{[L^2(S)]^2}.$$

Thus, $\mathbf{u}(\mathbf{x}) \to 0$ as $|x_1| \to \infty$ ($\mathbf{x} \in D_{h'}$), uniformly in x_2. From Lemma 2.5 and Theorem 2.7 it now follows that $J_j(A)[\mathbf{u}] \to 0$ as $A \to \infty$ ($j = 1, 2$), and consequently, by (4.23), that $\mathbf{Pu} = 0$ on S. Recalling once more Theorem 4.17, we conclude $\mathbf{u} \equiv 0$ in Ω.

We have thus shown the following central theorem:

Theorem 4.22 *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be solutions of Problem 4.15 with the same Dirichlet data* $\mathbf{g}$. *Then* $\mathbf{u} \equiv \mathbf{v}$ *in* Ω.

Chapter 5

Existence of Solution

Existence of solution will be proved by the boundary integral equation method, a well-established method to prove existence of solution for scattering problems. As a first step, an equivalent integral equation formulation of Problem 4.15 will be derived. However, as the integral operators in this equation are not compact, surjectivity cannot be deduced from injectivity by the Fredholm Alternative, as would be the usual approach in scattering by bounded obstacles. A new theory of solvability of integral equations on unbounded domains will be presented that allows this deduction for problems of scattering by infinite rough surfaces. This theory is used to establish solvability in $[BC(\mathbb{R})]^2$, first for the adjoint equation and then, through a duality argument, for the orginal equation. As a final result, it is shown how it is possible to establish solvability in all $[L^p(\mathbb{R})]^2$ spaces.

5.1 An Integral Equation Formulation

The first stepping stone in the existence proof will be an equivalent integral equation formulation of Problem 4.15. To derive this equation, we will seek a solution to Problem 4.15 in the form of a combined single- and double layer potential,

$$\mathbf{u}(\mathbf{x}) = \int_S \left\{ \Pi^{(2)}_{D,h}(\mathbf{x},\mathbf{y}) - i\eta\, \Gamma_{D,h}(\mathbf{x},\mathbf{y}) \right\} \psi(\mathbf{y})\, ds(\mathbf{y}), \qquad \mathbf{x} \in \Omega, \tag{5.1}$$

where $\psi \in [BC(S) \cap H^{1/2}_{loc}(S)]^2$ and η is a complex number with $\mathrm{Re}\,(\eta) > 0$. Throughout, we assume $\tilde{\mu} = \mu\,(\mu+\lambda)/(3\mu+\lambda)$ and $\tilde{\lambda} = (2\mu+\lambda)(\mu+\lambda)/(3\mu+\lambda)$ and that $f \in C^{1,1}(\mathbb{R})$.

From Theorem 3.1 we know that $\mathbf{u}$ is a solution to the Navier equation in Ω and in $U_h \setminus \bar{\Omega}$. Note also Remark 3.13 to see $\mathbf{u} \in [H^1_{loc}(\Omega)]^2$. Furthermore, it satisfies the UPRC by Theorem 4.14. A review of the derivation of Theorem 2.13 further reveals that the function $\mathcal{H}$ in this theorem satisfies $\mathcal{H}(x_2, y_2) = O(x_2)$ as $x_2 \to \infty$, uniformly for bounded y_2. Therefore, the growth condition (4.8) can be seen to hold with $\beta = -1/2$.

Thus, from the jump relations for elastic single- and double-layer potentials stated in Theorems 3.11 and 3.12 and the boundary condition, it follows that $\mathbf{u}$ is a solution to the boundary value problem if ψ is a solution to the integral equation

$$\frac{1}{2}\,\psi(\mathbf{x}) + \int_S \left\{ \Pi^{(2)}_{D,h}(\mathbf{x},\mathbf{y}) - i\eta\,\Gamma_{D,h}(\mathbf{x},\mathbf{y}) \right\} \psi(\mathbf{y})\, ds(\mathbf{y}) = -\mathbf{g}(\mathbf{x}), \qquad \mathbf{x} \in S. \tag{5.2}$$

It suffices to study the solvability of this integral equation in the space $[BC(S)]^2$, as it follows from Theorems 3.11 (c) and 3.12 (b) that the integral operator maps $[BC(S)]^2$ into $[H^{1/2}_{loc}(S)]^2$. Thus, for any right hand side $\mathbf{g} \in [BC(S) \cap H^{1/2}_{loc}(S)]^2$, the solution has to be an element of this space as well.

We now introduce the three integral operators $\mathbf{S}_f$, $\mathbf{D}_f$ and $\mathbf{D}'_f$ by

$$\mathbf{S}_f\,\phi(s) \;:=\; 2\int_{-\infty}^{\infty} \Gamma_{D,h}((s,f(s)),(t,f(t))\,\phi(t)\,\sqrt{1+f'(t)^2}\,dt, \tag{5.3}$$

$$\mathbf{D}_f\,\phi(s) \;:=\; 2\int_{-\infty}^{\infty} \Pi^{(2)}_{D,h}((s,f(s)),(t,f(t)))\,\phi(t)\,\sqrt{1+f'(t)^2}\,dt, \tag{5.4}$$

$$\mathbf{D}'_f\,\phi(s) \;:=\; 2\int_{-\infty}^{\infty} \Pi^{(1)}_{D,h}((s,f(s)),(t,f(t)))\,\phi(t)\,\sqrt{1+f'(t)^2}\,dt, \tag{5.5}$$

where $s \in \mathbb{R}$ and $\phi \in [L^\infty(\mathbb{R})]^2$. From Theorem 3.11 (c) and Theorem 3.12 (b), we know that each of these operators are bounded mappings of $[BC(\mathbb{R})]^2$ into $[C^{0,\alpha}(\mathbb{R})]^2$. Furthermore, setting $\phi(s) = \psi(s, f(s))$, the integral equation (5.2) is equivalent to

$$(\mathbf{I} + \mathbf{D}_f - i\eta\,\mathbf{S}_f)\,\phi(s) = -2\,\mathbf{g}(s,f(s)), \qquad s \in \mathbb{R}. \tag{5.6}$$

We explore the properties of these three integral operators more thoroughly in the following lemmas:

Lemma 5.1 *Let K denote any of the integral operators $\mathbf{S}_f$, $\mathbf{D}_f$ or $\mathbf{D}'_f$ and $\mathbf{K}(s,t)$ the corresponding matrix kernel, such that $K\phi(s) = \int_{-\infty}^{\infty} \mathbf{K}(s,t)\,\phi(t)\,dt$.*

(a) There exists a function $\kappa \in L^1(\mathbb{R})$ such that

$$\max_{j,k=1,2} |\mathbf{K}_{jk}(s,t)| \le \kappa(s-t) \qquad \textit{for almost all } s,t \in \mathbb{R},$$

where $\kappa = O(|s|^{-3/2})$ as $|s| \to \infty$.

(b) *The matrix kernel* $\mathbf{K}$ *of* K *satisfies the properties*

$$\sup_{\mathbf{s}\in\mathbb{R}} \int_{-\infty}^{\infty} |\mathbf{K}_{jk}(s,t)|\, dt < \infty, \qquad j,k = 1,2, \tag{5.7}$$

and, for all $s,\ s' \in \mathbb{R}$,

$$\lim_{s'\to s} \int_{-\infty}^{\infty} |\mathbf{K}_{jk}(s,t) - \mathbf{K}_{jk}(s',t)|\, dt = 0, \qquad j,k = 1,2. \tag{5.8}$$

(c) K *is a bounded mapping from* $[L^\infty(\mathbb{R})]^2$ *to* $[BC(\mathbb{R})]^2$.

(d) K *is a bounded mapping from* $[L^p(\mathbb{R})]^2$ *to* $[L^p(\mathbb{R})]^2$ *for any* $p \in [1,\infty)$.

Proof: For $s,\ t \in \mathbb{R}$, let $\mathbf{x} = (s, f(s))$, $\mathbf{y} = (t, f(t))$. Then, by (2.14) and Theorems 2.10 and 2.13, for $\varepsilon > 0$, $j,k = 1,2$, there exists a constant $C > 0$ and $\mathcal{H} \in C(\mathbb{R}^2)$ such that

$$|\Gamma_{D,h,jk}(\mathbf{x},\mathbf{y})| \leq \begin{cases} C\,(1 + \log|\mathbf{x}-\mathbf{y}|), & |s-t| \leq \varepsilon, \\ \frac{\mathcal{H}(f(s)-h, f(t)-h)}{|s-t|^{3/2}}, & |s-t| > \varepsilon. \end{cases}$$

The same estimate is valid for the fundamental matrices $\Pi_{D,h}^{(1)}$ and $\Pi_{D,h}^{(2)}$ by Theorem 2.16 and Remark 3.4. Thus (a) follows immediately.

Property (5.7) is a simple consequence of (a) while property (5.8) follows from the dominated convergence theorem. (5.7) and (5.8) together imply statement (c).

Statement (d) follows from (a) using Young's Theorem. ∎

Lemma 5.2 *The operator* $\mathbf{S}_f$ *is self-adjoint and the operators* $\mathbf{D}_f$ *and* $\mathbf{D}'_f$ *are adjoint with respect to the non-degenerate duality* $< \cdot,\cdot >$ *defined on* $[L^p(\mathbb{R})]^2 \times [L^q(\mathbb{R})]^2$, *by*

$$< \phi, \psi > := \int_{-\infty}^{\infty} \phi(s)\cdot\psi(s)\, \sqrt{1 + f'(s)^2}\, ds, \tag{5.9}$$

where either $1 < p,q < \infty$ *with* $1/p + 1/q = 1$ *or* $p = 1$ *and* $q = \infty$.

Proof: The assertion follows from Lemma 5.1 (d) as well as Lemma 2.15 and Theorem 2.16 (b). ∎

As the operators $\mathbf{S}_f$ and $\mathbf{D}_f$ are not compact on any suitable space, the Fredholm Alternative is not available as a tool to deduce surjectivity of $I + \mathbf{D}_f - i\eta\,\mathbf{S}_f$ from injectivity. We will nevertheless present a theory in the subsequent sections that allows this deduction.

However, it is also not straightforward to show injectivity of $I + \mathbf{D}_f - i\eta\,\mathbf{S}_f$. We will therefore start by considering the adjoint operator:

Theorem 5.3 *The operator* $I + \mathbf{D}'_f - i\eta\,\mathbf{S}_f$ *is injective on* $[L^\infty(\mathbb{R})]^2$.

Proof: Because of Lemma 5.1 (c), any vector density $\phi \in [L^\infty(\mathbb{R})]^2$ in the kernel of $I + \mathbf{D}'_f - i\eta\,\mathbf{S}_f$ has to be in $[BC(\mathbb{R})]^2$. It is therefore enough to show injectivity on this space.

Assume $\phi \in [BC(\mathbb{R})]^2$ to be a solution of

$$(I + \mathbf{D}'_f - i\eta\,\mathbf{S}_f)\phi = 0, \tag{5.10}$$

and define the single layer potential $\mathbf{u}$ by

$$\mathbf{u}(\mathbf{x}) := \int_S \Gamma_{D,h}(\mathbf{x},\mathbf{y})\,\phi(y_1)\,ds(\mathbf{y}) \qquad \text{for } \mathbf{x} \in U_h \setminus S,$$

where $h \in \mathbb{R}$ with $h < \inf f$. Let us introduce the notation $\mathbf{u}^+ = \mathbf{u}|_\Omega$ and $\mathbf{u}^- = \mathbf{u}|_{U_h \setminus \bar{\Omega}}$. From (5.10) and the mapping properties of $\mathbf{S}_f$ and $\mathbf{D}'_f$ it follows that $\phi \in [C^{0,\alpha}(\mathbb{R})]^2$. Thus, from Theorem 3.11, we have that $\mathbf{u}^-$ and $\mathbf{u}^+$ have continuous derivatives up to S and, moreover, that

$$\|\mathbf{u}^-\|_{1,\alpha;U_h\setminus\Omega} \le C\,\|\phi\|_{0,\alpha;S}, \tag{5.11}$$

$$\mathbf{P}\mathbf{u}^- - i\eta\,\mathbf{u}^- = 0 \qquad \text{on } S, \tag{5.12}$$

$$\phi = \mathbf{P}\mathbf{u}^- - \mathbf{P}\mathbf{u}^+. \tag{5.13}$$

Also, by the definition of $\Gamma_{D,h}$, there holds $\mathbf{u}^- = 0$ on T_h. Thus, by an application of the first generalised Betti formula,

$$\begin{aligned}\mathrm{Re}(\eta)\int_{S(A)} |\mathbf{u}^-|^2\,ds &= -\frac{1}{2i}\int_{S(A)} \left\{\mathbf{u}^- \cdot \mathbf{P}\overline{\mathbf{u}^-} - \overline{\mathbf{u}^-}\cdot \mathbf{P}\mathbf{u}^-\right\} ds \\ &= \frac{1}{2i}\int_{R(A)} \left\{\mathbf{u}^- \cdot \mathbf{P}\overline{\mathbf{u}^-} - \overline{\mathbf{u}^-}\cdot \mathbf{P}\mathbf{u}^-\right\} ds,\end{aligned} \tag{5.14}$$

where $R(A) := \{\mathbf{x} \in U_h \setminus \bar{\Omega} : |x_1| = A\}$. Note that the right hand side of this equation is bounded by (5.11), so $\mathbf{u}^- \in [L^2(S)]^2$ follows by taking the limit as $A \to \infty$. Consequently, (5.12) yields $\mathbf{P}\mathbf{u}^- \in [L^2(S)]^2$ as well.

It is not difficult to see that

$$\mathbf{u}^-(\mathbf{x}) = \int_S \left\{\Gamma_{D,h}(\mathbf{x},\mathbf{y})\,\mathbf{P}\mathbf{u}^-(\mathbf{y}) - \Pi^{(2)}_{D,h}(\mathbf{x},\mathbf{y})\,\mathbf{u}^-(\mathbf{y})\right\} ds(\mathbf{y})$$

holds for all $\mathbf{x} \in U_h \setminus \bar{\Omega}$. Thus, by an argument very similar to that in the uniqueness proof for the Dirichlet problem, it follows that $|\mathbf{u}^-(\mathbf{x})| \to 0$ as $|x_1| \to \infty$. Applying this in (5.14) and taking the limit for $A \to \infty$, then yields $\mathbf{u}^- = 0$ on S.

From the continuity of the elastic single-layer potential in U_h, there follows $\mathbf{u}^+ = 0$ on S and thus, by the uniqueness theorem for the Dirichlet problem, $\mathbf{P}\mathbf{u}^+ = 0$ on S. Also, (5.12) implies $\mathbf{P}\mathbf{u}^- = 0$ on S. Thus, by (5.13), we conclude $\phi \equiv 0$. ∎

5.2 Solvability results for a class of operator equations

To deduce the surjectivity of the integral operators from their injectivity, we will now present a solvability theory developed by Chandler-Wilde, Zhang and Ross [10,17,22] which generalises the collectively compact operator theory of Anselone [1]. The operator equation approach presented here is that used in [16].

Let X denote a Fréchet space with a countable family of semi-norms $\{|\cdot|_n\}$ that generates the topology on X. This topology shall be labelled τ. We define a subspace Y of X by

$$Y := \{\phi \in X : \|\phi\|_\infty := \sup_{n\in\mathbb{N}} |\phi|_n < \infty\}.$$

We note that $\|\cdot\|_\infty$ is a norm on Y and that Y becomes a Banach space under this norm. The corresponding topology shall be termed the *norm topology*.

Example 5.4 In the context of the application of this theory to the integral equation (5.6), we will use $X = [C(\mathbb{R})]^2$ with

$$|\phi|_n = \sup_{|t|\le n} |\phi(t)|,$$

so that $Y = [BC(\mathbb{R})]^2$.

On the space Y we now define a further topology as follows. Let S_0 denote the space of real sequences converging to 0,

$$S_0 := \{(a_n) : a_n \in \mathbb{R}, n \in \mathbb{N}, \lim_{n\to\infty} a_n = 0\}.$$

For each $a \in S_0$, we define a seminorm $|\cdot|_a$ on Y by

$$|\phi|_a := \sup_{n\in\mathbb{N}} |a_n \phi|_n.$$

Clearly, the family of seminorms $\{|\cdot|_a : a \in S_0\}$ is separating. Thus it induces a locally convex topology on Y, which we will call the σ-topology.

Remark 5.5 *The σ-topology is an abstraction of the* strict topology *introduced by* Buck *in [9].*

Remark 5.6 *For $(\phi_n) \subset Y$, we write $\phi_n \longrightarrow \phi$ if $\|\phi_n - \phi\|_\infty \to 0$ as $n \to \infty$, and $\phi_n \xrightarrow{\sigma} \phi$ if (ϕ_n) converges in the σ-topology. It is easy to see that*

$$\phi_n \xrightarrow{\sigma} \phi \quad \Longrightarrow \quad \|\phi\|_\infty \le \sup_{n\in\mathbb{N}} \|\phi_n\|_\infty. \tag{5.15}$$

The following theorem explores the properties of the σ-topology and its relation to the other two topologies.

Theorem 5.7

(a) The bounded sets in Y in the σ-topology and in the norm topology are the same.

(b) Y is complete in the σ-topology.

(c) A sequence $(\phi_n) \subset Y$ is convergent in the σ-topology if and only if it is convergent in the τ-topology and (ϕ_n) is bounded in the norm topology.

(d) The σ-topology is either identical to the norm topology or it is not metrizable.

(e) If $K : Y \to Y$ is a linear operator and K is bounded in the σ-topology then it is also continuous in the norm topology.

Proof: Obviously any bounded set in the norm topology is bounded in the σ-topology. Assume $B \subset Y$ to be bounded in the σ-topology but not in the norm topology. Then, there exists a sequence $(\phi_n) \subset B$ and a sequence $(a_n) \in S_0$ such that $|\phi|_n \geq a_n^{-2}$. But then,

$$|a_n \, \phi|_n \geq \frac{1}{|a_n|} \longrightarrow \infty, \qquad n \to \infty,$$

which is a contradiction. Thus (a) follows.

Next, we prove (c). Let $(\phi_n) \subset Y$ be convergent with limit $\phi \in Y$ in the σ-topology. Then clearly, (ϕ_n) is bounded in the σ-topology and thus also bounded in the norm topology by (a). If, for $j \in \mathbb{N}$, we choose $a \in S_0$ such that $a_j = 1$, then there holds

$$|\psi|_j \leq |\psi|_a, \qquad \psi \in Y. \tag{5.16}$$

Thus $\phi_n \to \phi$ in the τ-topology as well. Conversely, assume $(\phi_n) \subset Y$ to be bounded in the norm topology and convergent to $\phi \in X$ in the τ-topology. Let $M := \sup_{n \in \mathbb{N}} \|\phi_n\|_\infty$. Then, for every $k \in \mathbb{N}$ there exists $n(k) \in \mathbb{N}$ such that $|\phi - \phi_{n(k)}|_k \leq M$. Thus,

$$|\phi|_k \leq |\phi - \phi_{n(k)}|_k + |\phi_{n(k)}|_k \leq 2\, M,$$

and consequently $\phi \in Y$. Therefore, now $\phi_n \stackrel{\sigma}{\longrightarrow} \phi$.

To prove (b), assume $(\phi_n) \subset Y$ to be a Cauchy sequence in the σ-topology. Then, by (a), it is bounded in the norm topology and by (5.16), (ϕ_n) is also a Cauchy sequence in X with respect to the τ-topology and thus convergent. Therefore, (c) now implies that (ϕ_n) is convergent in the σ-topology.

To prove (d), assume that the σ-topology is metrizable. Then, by Baire's Theorem [43, Theorem 2.2], (Y,σ) is of second category in itself and thus, by the Open Mapping Theorem [43, Theorem 2.11], the mapping $\mathrm{id} : (Y, \|\cdot\|_\infty) \to (Y, \sigma)$ is open. But the σ-topology is weaker than the norm topology so both must be identical.

By (a), an operator K bounded in the σ-topology is also bounded in the norm topology. As boundedness is equivalent to continuity on normed spaces, (e) follows. ■

Definition 5.8 *A set $B \subset Y$ is said to be* relatively sequentially compact in the σ-topology *if any sequence in B has a subsequence that is convergent in the σ-topology. A linear operator K is said to be* relatively sequentially compact with respect to (w.r.t.) the σ-topology *if for any bounded set $B \subset Y$, $K(B)$ is relatively sequentially compact in the σ-topology. A family $\mathcal{K}$ of linear operators on Y is said to be* collectively sequentially compact w.r.t. the σ-topology *if for any bounded set $B \subset Y$ the set $\bigcup_{K\in\mathcal{K}} K(B)$ is relatively sequentially compact in the σ-topology.*

In some of the subsequent arguments we will also make use of the following notion:

Definition 5.9 *An operator $L : Y \to Y$ is said to be σ-norm-continuous, if it is continuous as an operator from (Y, σ) to $(Y, \|\cdot\|)$.*

We also denote, as is usual, the set of Fredholm operators on a given topological vector space V by $\Phi(V)$.

The following Lemma, in a less general form, is due to Haseloh [32, 33] and in the proof we follow the original argument.

Lemma 5.10 *Let K, L be sequentially compact w.r.t the σ-topology and assume additionally that L is σ-norm-continuous.*

(a) *LK is compact.*

(b) *$I - L \in \Phi(Y)$ with $\mathrm{ind}\,(I - L) = 0$.*

(c) *$I - K - L \in \Phi(Y)$ if and only if $I - K \in \Phi(Y)$. If one of these statements hold then the index of both operators is the same.*

Proof: To see (a), let $(\phi_n) \subset Y$ be bounded. Then $(K\phi_n)$ has a σ-convergent subsequence. The image of this subsequence under L is convergent in Y with respect to the norm topology.

Next we prove (b). There holds $(I+L)(I-L) = I - L^2$, and L^2 is compact by (a). Thus $I - L \in \Phi(Y)$ follows from standard operator theory. Moreover, the same holds for $I - \alpha L$ with any $\alpha \in [0,1]$ and the indices of all these operators are the same, and thus identical to $\operatorname{ind} I = 0$.

To see (c), we suppose that $I - K \in \Phi(Y)$. As $(I-L)(I-K) = I - L - K + LK$, we have, by Atkinson's Theorem and also because LK is compact and $\operatorname{ind}(I-L) = 0$, that $I - L - K \in \Phi(Y)$ with $\operatorname{ind}(I-L-K) = \operatorname{ind}(I-K)$. Note that this argument can be reversed. ∎

For some of the arguments developed later, it is also necessary to develop a notion of the convergence of operators.

Definition 5.11 *Assume K_n, K to be linear operators in Y, $n \in \mathbb{N}$. We will write $K_n \xrightarrow{\sigma} K$ if*

$$\phi_n \xrightarrow{\sigma} \phi \quad \Longrightarrow \quad K_n\phi_n \xrightarrow{\sigma} K\phi.$$

Finally, let $\mathcal{I}$ be a set of isometries on Y. $\mathcal{I}$ will be said to be sufficient if, for some $j \in \mathbb{N}$ and for each $\phi \in Y$, there exists $J \in \mathcal{I}$ such that $|J\phi|_j \geq (1/2)\,\|\phi\|_\infty$.

We have now collected all the building blocks necessary for the proof of a uniform bound for the inverse of operators $I - K$ with K in certain collectively sequentially compact families of operators:

Theorem 5.12 *Suppose that $\mathcal{K} \subset B(Y)$ is collectively sequentially compact with respect to the σ-topology and that for every sequence $(K_n) \subset \mathcal{K}$ there exists a subsequence (K_{n_m}) and $K \in \mathcal{K}$ such that $K_{n_m} \xrightarrow{\sigma} K$. Further suppose that $\mathcal{I}$ is a sufficient set of isometries on Y and that for all $n \in \mathbb{N}$, $K \in \mathcal{K}$ and $J \in \mathcal{I}$ there holds $JKJ^{-1} \in \mathcal{K}$. Lastly, assume that $I - K$ is injective for all $K \in \mathcal{K}$. Then,*

$$\sup_{K \in \mathcal{K}} \|(I-K)^{-1}\| < \infty.$$

Proof: Assume that the theorem is not true. Then there exists a sequence $(\phi_n) \subset Y$ with $\|\phi_n\|_\infty = 1$ and a sequence $(K_n) \subset \mathcal{K}$ with

$$\|(I - K_n)\,\phi_n\|_\infty \longrightarrow 0, \qquad n \to \infty.$$

For some $j \in \mathbb{N}$ and for each ϕ_n, there exists $J_n \in \mathcal{I}$ such that $|J_n \phi_n|_j \geq 1/2$. Observing

$$(I - K_n)\,\phi_n = J_n^{-1}\,(I - J_n K_n J_n^{-1})\,J_n\phi_n,$$

we set $\tilde{K}_n := J_n K_n J_n^{-1}$ and $\psi_n = J_n\phi_n$ and can assume, by using appropriate subsequences, that there exist $\tilde{K} \in \mathcal{K}$ and $\psi \in Y$ such that $\tilde{K}_n \xrightarrow{\sigma} \tilde{K}$ and $\tilde{K}_n\psi_n \xrightarrow{\sigma} \psi$. Note that also $\psi_n - \tilde{K}_n\psi_n \xrightarrow{\sigma} 0$. Thus,

$$\psi_n = \psi - (\psi - \tilde{K}_n\psi_n) - (\tilde{K}_n\psi_n - \psi_n),$$

and by taking limits in the σ-topology it follows that $\psi_n \xrightarrow{\sigma} \psi$. Since $\tilde{K}_n \xrightarrow{\sigma} \tilde{K}$, it follows also that $\tilde{K}_n\psi_n \xrightarrow{\sigma} \tilde{K}\psi$. We now conclude that $\psi - \tilde{K}\psi = 0$ and, as $\tilde{K} \in \mathcal{K}$ is injective, that $\psi = 0$. On the other hand, $|\psi_n|_j \geq 1/2$ for all n, which is a contradiction. ■

The next theorem will establish conditions, additional to those in Theorem 5.12, which will ensure that $I - K$ is also surjective, so that $(I - K)^{-1}$ is a bounded operator in Y for all $K \in \mathcal{K}$.

Theorem 5.13 *Suppose that all the conditions of Theorem 5.12 are satisfied and that, in addition, for every $K \in \mathcal{K}$, there exists a sequence $(K_n) \subset \mathcal{K}$ such that $I - K_n$ is bijective, $n \in \mathbb{N}$, and $K_n \xrightarrow{\sigma} K$, then $I - K$ is bijective for every $K \in \mathcal{K}$.*

Proof: Let $\psi \in Y$ and $K \in \mathcal{K}$. Then there exists $(K_n) \subset \mathcal{K}$ with $I - K_n$ bijective and $K_n \xrightarrow{\sigma} K$. Set $\phi_n := (I - K_n)^{-1}\psi$. As (ϕ_n) is bounded, the sequence $(K_n\phi_n)$ has a σ-converging subsequence $K_{n_m}\phi_{n_m}$. Choose $\phi \in Y$ such that

$$K_{n_m}\phi_{n_m} \xrightarrow{\sigma} \phi - \psi, \qquad m \to \infty.$$

Then $\phi_{n_m} = K_{n_m}\phi_{n_m} + \psi \xrightarrow{\sigma} \phi$ and thus $K_{n_m}\phi_{n_m} \xrightarrow{\sigma} K\phi$. Combining the limits for $K_{n_m}\phi_{n_m}$ and for ϕ_{n_m} now yields $\phi - K\phi = \psi$. Thus $I - K$ is surjective. ■

5.3 Solvability in $[BC(\mathbb{R})]^2$

We will now tackle the task of applying the results of the preceding section to deduce surjectivity from injectivity for the operators $I + \mathbf{D}_f - i\eta\,\mathbf{S}_f$ and $I + \mathbf{D}'_f - i\eta\,\mathbf{S}_f$. To this end it is necessary to provide the framework in which to prove that the assumptions of Theorems 5.12 and 5.13 are satisfied.

We recall Example 5.4 for the relevant definitions of X and the family of semi-norms $\{|\cdot|_n\}$. We now introduce, for $c > h$, $M > 0$, the set

$$B_{1,c,M} := \{f \in C^{1,1}(\mathbb{R}) : \|f\|_{1,1;\mathbb{R}} \leq M, \min f > c\}$$

and the families of operators

$$\begin{aligned} \mathcal{K}_1 &:= \{i\eta\,\mathbf{S}_f - \mathbf{D}_f : f \in B_{1,c,M}\}, \\ \mathcal{K}_2 &:= \{i\eta\,\mathbf{S}_f - \mathbf{D}'_f : f \in B_{1,c,M}\}. \end{aligned}$$

For notational purposes, we note that any integral operator $K_f \in \mathcal{K}_j$, $j = 1, 2$ can be written in the form

$$K_f\,\phi(s) := \int_{-\infty}^{\infty} \mathbf{K}_f(s,t)\,\phi(t)\,dt,$$

with some matrix kernel $\mathbf{K}_f$. This notation will be used throughout the following arguments.

The following Corollary is a consequence of Lemma 5.1:

Corollary 5.14 *Let $K_f \in \mathcal{K}_j$, $j = 1, 2$. Then K_f is continuous and sequentially compact w.r.t. the σ-topology.*

Proof: The proof of the corollary is identical to that of [32, Theorem 1.2.1] and [33, Satz 2.16]. If $\phi_n \xrightarrow{\sigma} \phi$, then (ϕ_n) is bounded by, say, M and thus

$$|K_f\,(\phi_n - \phi)(s)| \leq \int_{-A}^{A} |\mathbf{K}(s,t)\,(\phi_n(t) - \phi(t))|\,dt + 4\,M \int_{|t|>A} |\kappa(s-t)|\,dt$$

for any $s \in \mathbb{R}$ and $A > 0$, where κ is the function in Lemma 5.1 (a). Given $\varepsilon > 0$, the second term on the right hand side can be made smaller than $\varepsilon/2$ by choosing A large enough, and, for fixed A, the first term will also be smaller then $\varepsilon/2$ for s in any compact set, for any n large enough. This shows that K_f is continuous w.r.t. the σ-topology.

That K_f is sequentially compact w.r.t. the σ-topology follows from the fact that K_f maps bounded sets into bounded, equicontinuous sets (see e.g. [2, 41] for details). ■

To formulate the subsequent Lemmas more concisely, we adopt the following notation for uniform convergence of sequences in $BC(\mathbb{R})$ on finite intervals:

$$f_n \xrightarrow{s} f \quad \text{iff} \quad \left\{ \begin{array}{c} (f_n) \text{ bounded in } BC(\mathbb{R}), \\ \|f - f_n\|_{\infty;I} \to 0 \ (n \to \infty) \text{ for any } I \subset\subset \mathbb{R}. \end{array} \right.$$

We now have all the necessary means to prove the assumptions made in the previous section for the case of the operator families $\mathcal{K}_j$:

Lemma 5.15

(a) *For every sequence* $(f_n) \subset B_{1,c,M}$ *there exists a subsequence* (f_{n_m}) *and* $f \in B_{1,c,M}$ *such that* $f_{n_m} \xrightarrow{s} f$, $f'_{n_m} \xrightarrow{s} f'$.

(b) *Let* $(K_{f_n}) \subset \mathcal{K}_j$ *and* $K_f \in \mathcal{K}_j$, $j = 1, 2$, *such that* $f_n \xrightarrow{s} f$, $f'_n \xrightarrow{s} f'$. *Then, for any* $\varepsilon > 0$,

$$\sup_{(s,t)\in D_\varepsilon} |\mathbf{K}_{f_n,jk}(s,t) - \mathbf{K}_{f,jk}(s,t)| \longrightarrow 0, \qquad n \to \infty, \qquad j, k = 1, 2,$$

where $D_\varepsilon := \{(s,t) \in D \times \mathbb{R} : \varepsilon < |s - t| < 1/\varepsilon\}$ *and* $D \subset\subset \mathbb{R}$.

Proof: Statement (a) is proven in [15, Lemma 4.4 (i)]. Statement (b) follows from the regularity of the fundamental matrices $\Gamma_{D,h}$, $\Pi^{(j)}_{D,h}$, $j = 1, 2$, on the compact set

$$\Omega_\varepsilon = \{(\mathbf{x}, \mathbf{y}) : \mathbf{x} = (s, f_n(s)), \mathbf{y} = (t, f_n(t)), (s,t) \in D_\varepsilon, f_n \in B_{1,c,M}\}.$$

■

Lemma 5.16 *Assume that* $(\phi_n) \subset [BC(\mathbb{R})]^2$ *is a bounded sequence and that there is a sequence* $(K_{f_n}) \subset \mathcal{K}_j$ *and an operator* $K_f \in \mathcal{K}_j$, $j = 1, 2$ *such that* $f_n \xrightarrow{s} f$, $f'_n \xrightarrow{s} f'$. *Then* $(K_{f_n} - K_f)\phi_n \xrightarrow{\sigma} 0$.

Proof: Assume $D \subset\subset \mathbb{R}$ and $s \in D$. Then

$$\begin{aligned} |(K_{f_n} - K_f)\,\phi_n(s)| \;\; &\leq \;\; 2\,\|\phi_n\|_\infty \Big\{ \max_{j,k=1,2} \int\limits_{\varepsilon<|s-t|<1/\varepsilon} |\mathbf{K}_{f_n,jk}(s,t) - \mathbf{K}_{f,jk}(s,t)|\, dt \\ &\qquad + \max_{j,k=1,2} \int\limits_{\substack{|s-t|\leq\varepsilon \\ \text{or } |s-t|\geq 1/\varepsilon}} |\mathbf{K}_{f_n,jk}(s,t) - \mathbf{K}_{f,jk}(s,t)|\, dt \Big\}. \end{aligned}$$

Letting $n \to \infty$, the first integral vanishes by Lemma 5.15 (b) while the second is seen to be bounded independently of n and $s \in D$. Letting now $\varepsilon \to 0$ shows that this bound tends to 0. It is also not difficult to see that the sequence $((K_{f_n} - K_f)\phi_n)$ is bounded, so the assertion follows by Theorem 5.7 (c). ■

Theorem 5.17 *The sets* $\mathcal{K}_j$, $j = 1, 2$, *are collectively sequentially compact with respect to the* σ*-topology. Furthermore, for every sequence* $(K_{f_n}) \subset \mathcal{K}_j$, *there exists a subsequence* $(K_{f_{n_m}})$ *and* $K_f \in \mathcal{K}_j$ *such that* $K_{f_{n_m}} \xrightarrow{\sigma} K_f$.

Proof: Let $B \subset [BC(\mathbb{R})]^2$ be bounded and $(\psi_n) \subset \bigcup_{K_f \in \mathcal{K}_j} K_f(B)$, $j = 1, 2$. We choose $(K_{f_n}) \subset \mathcal{K}_j$ and $(\phi_n) \subset B$ with $\psi_n = K_{f_n}\phi_n$. By Lemma 5.15 (a), there exists a subsequence (f_{n_m}) and $f \in B_{1,c,M}$ such that $f_{n_m} \xrightarrow{s} f$, $f'_{n_m} \xrightarrow{s} f'$. The sequence (ϕ_{n_m}) is bounded, so by Corollary 5.14 we can choose a subsequence, denoted for notational simplicity by (ϕ_k), such that, with $K_f \in \mathcal{K}_j$, $K_f\phi_k \xrightarrow{\sigma} \psi \in [BC(\mathbb{R})]^2$. But $K_{f_k}\phi_k = K_f\phi_k + (K_{f_k} - K_f)\phi_k$, and the second term in this sum converges to 0 in the σ-topology by Lemma 5.16. Thus $K_{f_k}\phi_k \xrightarrow{\sigma} \psi$. The second assertion of the theorem is proved by very similar arguments and also the fact that K_f is continuous w.r.t. the σ-topology. ■

As the next step in the argument, we now introduce the set of translation operators $\mathcal{T}$:

$$\mathcal{T} := \{T_a : [BC(\mathbb{R})]^2 \to [BC(\mathbb{R})]^2, \phi \mapsto \phi(\cdot - a),\ a \in \mathbb{R}.\}$$

Obviously, $\mathcal{T}$ forms a sufficient subgroup of the group of isometries on $[BC(\mathbb{R})]^2$. Furthermore, as for $f \in B_{1,c,M}$ there also holds $f(\cdot - a) \in B_{1,c,M}$, it is not difficult to see that for $K_f \in \mathcal{K}_j$, $T_a \in \mathcal{T}$, there holds $T_{-a}KT_a \in \mathcal{K}_j$.

Let now $K_f \in \mathcal{K}_j$, $j = 1, 2$, and denote by χ a $C^\infty(\mathbb{R})$ function with $|\chi| \le 1$ on $\mathbb{R}$, $\chi(t) = 0$ for $t \le 0$ and $\chi(t) = 1$ for $t \ge 1$, we define

$$\chi_n(t) := \begin{cases} \chi(t + n + 1), & t < 0, \\ \chi(n + 1 - t), & t \ge 0, \end{cases} \qquad n \in \mathbb{N}.$$

Now, also setting $\bar{f} := (\sup f + \inf f)/2$, we construct the sequence $(f_n) \subset B_{1,c,M}$ by

$$f_n(t) := \chi_n(t)\, f(t) + (1 - \chi_n(t))\, \bar{f}.$$

Then $f_n \xrightarrow{s} f$, $f'_n \xrightarrow{s} f'$ follows, and using Lemma 5.16, for $K_{f_n} \in \mathcal{K}_j$ also that $K_{f_n} \xrightarrow{\sigma} K_f$.

We now split up the integral operators K_{f_n} by

$$K_{f_n} = K_{\bar{f}} + L_{f_n},$$

where

$$L_{f_n}\phi(s) := \int_{-n-1}^{n+1} (\mathbf{K}_{f_n}(s,t) - \mathbf{K}_{\bar{f}}(s,t))\, \phi(t)\, dt.$$

Lemma 5.18 *The operators L_{f_n}, $n \in \mathbb{N}$, are compact.*

Proof: We have that

$$\max_{j,k=1,2} \int_{-n-1}^{n+1} |\mathbf{K}_{f_n,jk}(s,t) - \mathbf{K}_{\bar{f},jk}(s,t))|\, dt \le \int_{-n-1}^{n+1} \kappa_{f_n}(s-t) + \kappa_{\bar{f}}(s-t)\, dt,$$

where κ_{f_n} and $\kappa_{\bar{f}}$ denote the functions in Lemma 5.1, respectively. Thus, each component of the matrix kernel of L_{f_n} satisfies condition **C** of [12]. Also observing that conditions (5.7) and (5.8) hold for L_{f_n}, we can apply [12, Lemma 2.1] to obtain the assertion (see also [2,3] for details on this argument). ■

Summing up the results so far, now yields a first solvability theorem for the integral equations in question:

Theorem 5.19 *Assume $I - K_f$ to be injective for all $K_f \in \mathcal{K}_j$, $j = 1, 2$. Then the operators $I - K_f$ are all bijective and there holds*

$$\sup_{K_f \in \mathcal{K}_j} \|(I - K_f)^{-1}\| < \infty.$$

Proof: The first part of the assertion follows from Theorem 5.12, also using Corollary 5.14, Theorem 5.17 and the subsequent remarks on translation operators.

For a given $K_f \in \mathcal{K}_j$, we now construct sequences (f_n) and (K_{f_n}) as indicated above. The operator $K_{\bar{f}}$ is now a convolution operator with a kernel in $[L^1(\mathbb{R})]^{2\times 2}$. Thus, we can apply Theorem A.2 in [18] to see that $I - K_{\bar{f}}$ is bijective and thus a Fredholm operator of index 0 on $[BC(\mathbb{R})]^2$. However, L_{f_n} is compact by Lemma 5.18, so $I - K_{f_n}$ is also Fredholm with index 0, and thus has to be bijective as well. We can now apply Theorem 5.13 to obtain finally that $I - K_f$ is surjective. ■

Recalling Theorem 5.3, we immediately obtain the following result for operators in $\mathcal{K}_2$:

Corollary 5.20 *For the set $\mathcal{K}_2$, there holds*

$$\sup_{K_f \in \mathcal{K}_2} \|(I - K_f)^{-1}\| < \infty.$$

Furthermore, for every $K_f \in \mathcal{K}_2$, $I - K_f$ is surjective.

5.4 Weighted Spaces

The previous section has yielded solvability results for the operator family $\mathcal{K}_2$ in $[BC(\mathbb{R})]^2$. To obtain similar results for $\mathcal{K}_1$, it is necessary to study solvability for operators in $\mathcal{K}_2$ in different spaces.

Let us introduce the weight functions w_p, defined by

$$w_p(s) = (1 + |s|)^p, \qquad p \in \mathbb{R}.$$

We can define subsets of $Y = [BC(\mathbb{R})]^2$ by

$$Y_p := \{\phi \in Y : \|w_p\,\phi\|_\infty < \infty\}, \qquad p > 0.$$

The space Y_p becomes a Banach space with the norm $\|\cdot\|_p$ defined by $\|\phi\|_p := \|w_p\,\phi\|_\infty$. The operator W_p of premultiplication by w_p is an isometric isomorphism from Y_p to Y. Its inverse W_{-p} is an isometric isomorphism from Y to Y_p.

In the following arguments, for any normed space E, let $\mathcal{B}(E)$ denote the space of bounded linear operators in E. Assume now that $K \in \mathcal{B}(Y)$ is such that also $K \in \mathcal{B}(Y_p)$ for $p \in [0, q]$ for some $q > 0$. We will also assume that K is sequentially compact w.r.t. the σ-topology. Define

$$K^{(p)} := W_p\,K\,W_{-p}, \qquad p > 0.$$

Theorem 5.21 *Suppose that $K - K^{(p)}$ is sequentially compact w.r.t. the σ-topology and also σ-norm-continuous for all $p \in [0, q]$. Further suppose that $I - K \in \Phi(Y_p)$ for some $p \in [0, q]$. Then $I - K \in \Phi(Y_{p'})$ for all $p' \in [0, q]$ and the index of $I - K$ is the same in all these spaces.*

Proof: From the definition of $K^{(p)}$ and Atkinson's Theorem it follows that

$$I - K \in \Phi(Y_p) \Longleftrightarrow I - K^{(p)} \in \Phi(Y),$$

and that, if one of these statements is true, the indices of both operators are the same. Now suppose $I - K \in \Phi(Y_p)$ for some $p \in [0, q]$. We have $I - K = I - K^{(p)} - (K - K^{(p)})$, so it follows from Lemma 5.10 and the equivalence above that $I - K \in \Phi(Y)$ and $\operatorname{ind}(I - K) = \operatorname{ind}(I - K^{(p)})$ in $\Phi(Y)$. Reversing this argument yields $I - K^{(p')} \in \Phi(Y)$ for all $p' \in [0, q]$ and thus the assertion. ∎

In the following lemma we will study how Theorem 5.21 can be applied to an operator $K \in \mathcal{K}_2$.

Lemma 5.22 *Assume $K_f \in \mathcal{K}_2$ and $p \in [0, 3/2)$. Then,*

(a) $K_f \in \mathcal{B}(Y_p)$,

(b) $K_f - K_f^{(p)}$ *is sequentially compact w.r.t. the σ-topology and also σ-norm-continuous,*

(c) $I - K_f \in \Phi(Y_p)$ *and* $\operatorname{ind}(I - K_f) = 0$. *Thus $I - K_f$ is bijective as an operator on Y_p.*

Proof: From Lemma 5.1 (a) and [40, Theorem 3.5] it follows that $\mathbf{K}^{(p)}_{f,jk}$ satisfies the properties (5.7) and (5.8) which in turn yields $K^{(p)}_f \in \mathcal{B}(Y)$. This last statement is obviously equivalent to $K_f \in \mathcal{B}(Y_p)$.

As now both $\mathbf{K}_f$ and $\mathbf{K}^{(p)}_f$ satisfy properties (5.7) and (5.8), it follows that $K_f - K^{(p)}_f$ is sequentially compact w.r.t. the σ-topology. Theorem 3.6 in [40] also yields that $K_f - K^{(p)}_f$ is compact in Y, so an application of Theorem 1.2.4 and Lemma 1.2.3 in [32] yields that $K_f - K^{(p)}_f$ is σ-norm-continuous.

We are now in a position to apply Theorem 5.21. From Theorems 5.3 and 5.20 we know that $I - K_f \in \Phi(Y)$ with ind $(I - K_f) = 0$. Thus it follows that $I - K_f \in \Phi(Y_p)$ with the same index. As it is an injective operator by Theorem 5.3, it follows that $I - K_f$ is in fact bijective on Y_p. ∎

Using the previous Lemma and the solvability theory developed in the previous section now immediately yields the following corollary:

Corollary 5.23 *For any $K_f \in \mathcal{K}_1$, the operator $I - K_f$ is bijective on $[BC(\mathbb{R})]^2$ and there holds*

$$\sup_{K_f \in \mathcal{K}_1} \|(I - K_f)^{-1}\| \leq \infty.$$

Proof: Given $K_f \in \mathcal{K}_1$, we know that $K'_f \in \mathcal{K}_2$ and thus $I - K'_f$ is bijective on $[BC(\mathbb{R})]^2$. Thus, by Lemma 5.22 it is also bijective on Y_p for any $1 < p < 3/2$. On the other hand, Y_p is dense in $[L^1(\mathbb{R})]^2$, so using a standard duality argument, it follows that $I - K_f$ is injective on $[L^\infty(\mathbb{R})]^2$ and hence also on Y.

An application of Theorem 5.19 now yields that $I - K_f$ is bijective on Y for all $K_f \in \mathcal{K}_1$ and that the inverse operators are uniformly bounded. ∎

In terms of the original formulation of the scattering problem as a boundary value problem, Problem 4.15, we can now answer the question of existence of solution in the affirmative:

Theorem 5.24 *For any Dirichlet data $\mathbf{g} \in [BC(S) \cap H^{1/2}_{loc}(S)]^2$, there exists a uniquely determined solution $\mathbf{u} \in [C^2(\Omega) \cap C(\bar{\Omega}) \cap H^1_{loc}(\Omega)]^2$ to Problem 4.15. Moreover, the solution $\mathbf{u}$ depends continuously on $\|\mathbf{g}\|_{\infty;S}$, uniformly in $[C(\bar{\Omega} \setminus U_H)]^2$ for any $H > \sup f$.*

5.5 Solvability in $[L^p(\mathbb{R})]^2$

The solvability results derived so far can be extended in a very interesting fashion; it is in fact possible to prove solvability of equation (5.6) in all $[L^p(\mathbb{R})]^2$ spaces,

$p \in [1, \infty]$. Let us start with the simplest case, $[L^\infty(\mathbb{R})]^2$.

Lemma 5.25 *Assume K to be a bounded linear operator in $[L^\infty(\mathbb{R})]^2$ such that $I - K$ is bijective as an operator on $[BC(\mathbb{R})]^2$ and its inverse is bounded. Further assume that $K([L^\infty(\mathbb{R})]^2) \subset [BC(\mathbb{R})]^2$. Then $I - K$ is bijective on $[L^\infty(\mathbb{R})]^2$ and its inverse is bounded, with*

$$\|(I-K)^{-1}\|_{[L^\infty(\mathbb{R})]^2 \to [L^\infty(\mathbb{R})]^2} \leq 1 + \|(I+K)^{-1}\|_{[BC(\mathbb{R})]^2 \to [BC(\mathbb{R})]^2} \, \|K\|_{[L^\infty(\mathbb{R})]^2 \to [BC(\mathbb{R})]^2}.$$

Proof: It was already pointed out in the proof of Theorem 5.3 that injectivity of $I - K$ on $[BC(\mathbb{R})]^2$ implies injectivity on $[L^\infty(\mathbb{R})]^2$.

Now assume $\psi \in [L^\infty(\mathbb{R})]^2$. Then $K\psi \in [BC(\mathbb{R})]^2$, so that there exists $\chi \in [BC(\mathbb{R})]^2$ with $(I+K)\,\chi = -K\psi$. Set $\phi = \psi + \chi$. Then $(I+K)\phi = \psi$ follows. Moreover, there holds

$$\|\phi\|_{[L^\infty(\mathbb{R})]^2} \leq \|\psi\|_{[L^\infty(\mathbb{R})]^2} (1 + \|(I+K)^{-1}\|_{[BC(\mathbb{R})]^2 \to [BC(\mathbb{R})]^2} \, \|K\|_{[L^\infty(\mathbb{R})]^2 \to [BC(\mathbb{R})]^2}).$$

■

Corollary 5.26 *For $K_f \in \mathcal{K}_j$, $j = 1, 2$, $I - K_f$ is bijective as an operator on $[L^\infty(\mathbb{R})]^2$, its inverse is bounded and*

$$\sup_{K_f \in \mathcal{K}_j} \|(I - K_f)^{-1}\| < \infty.$$

Proof: The assertion follows immediately from the previous lemma, also observing the estimate for its inverse. ■

We will now consider $K_f \in \mathcal{K}_1$ as an operator on $[L^1(\mathbb{R})]^2$. Making use of the norm-isomorphism $I_f : [L^1(\mathbb{R})]^2 \to [L^1(S)]^2$ defined by

$$I_f \phi(\mathbf{x}) := \phi(x_1) \qquad \mathbf{x} \in S,$$

we obtain the operator $K := I_f \, K_f \, I_f^{-1}$ on $[L^1(S)]^2$. In the same way we obtain from $K_f' \in \mathcal{K}_2$ as an operator on $[L^\infty(\mathbb{R})]^2$ the operator K' on $[L^\infty(S)]^2$. Note that

$$\|I_f\| \leq (1 + \|f'\|^2_{\infty;\mathbb{R}})^{1/2} \qquad \text{and} \qquad \|I_f^{-1}\| \leq 1,$$

so that these operators are uniformly bounded for all $f \in B_{1,c,M}$. Note also that the dual operator K^* of K is given by

$$K^* \psi = \overline{K' \overline{\psi}}, \qquad \psi \in [L^\infty(S)]^2.$$

As $I - K_f'$ is bijective on $[L^\infty(\mathbb{R})]^2$, the same holds for $I - K^*$ on $[L^\infty(S)]^2$. Thus we can apply [43, Theorem 4.12] to obtain that $I - K$ is injective on $[L^1(S)]^2$ and its range is dense. Moreover, [43, Theorem 4.14] states that the range of $I - K$ is closed in $[L^1(S)]^2$. For ϕ, $\psi \in [L^1(S)]^2$ satisfying $(I - K)\,\phi = \psi$, we also obtain, using the Cauchy-Schwarz inequality, that

$$\|\phi\| = \sup_{\|\phi^*\| \leq 1} | < \phi, \phi^* > | \leq \|\psi\| \, \|(I - K^*)^{-1}\|.$$

Applying these results to the operator $K_f \in \mathcal{K}_1$ and also observing that the same argument can be applied to $K_f \in \mathcal{K}_2$, we immediately have the following theorem:

Theorem 5.27 *Assume $K_f \in \mathcal{K}_j$, $j = 1, 2$. Then $I - K_f$ is bijective as an operator on $[L^1(\mathbb{R})]^2$, its inverse is bounded and*

$$\sup_{K_f \in \mathcal{K}_j} \|(I - K_f)^{-1}\| < \infty.$$

For the final step in our argument, we will now make use of an interpolation theorem due to M. Riesz and Thorin which is also called *Riesz Convexity Theorem*. It can be found, in more general form than stated here, in [45, Chapter V, Theorem 1.3] and also in [7].

Theorem 5.28 (Riesz-Thorin) *Let L denote a linear operator that is bounded as a mapping from $[L^1(\mathbb{R})]^2$ to $[L^1(\mathbb{R})]^2$ with norm N_1 and as a mapping from $[L^\infty(\mathbb{R})]^2$ to $[L^\infty(\mathbb{R})]^2$ with norm N_∞. Then L is also a bounded mapping from $[L^p(\mathbb{R})]^2$ to $[L^p(\mathbb{R})]^2$, $p \in (1, \infty)$, with norm N_p, where*

$$N_p \leq N_1^{\frac{1}{p}} \, N_\infty^{1-\frac{1}{p}}.$$

Applying Theorem 5.28 to the operator $I - K_f$ with $K_f \in K_j$, $j = 1, 2$, we obtain the following general solvability result for the integral equation (5.6):

Theorem 5.29 *Assume $K_f \in \mathcal{K}_j$, $j = 1, 2$. Then $I - K_f$ is bijective as an operator on $[L^p(\mathbb{R})]^2$, $p \in [1, \infty) \cup \{\infty\}$, its inverse is bounded and*

$$\sup_{K_f \in \mathcal{K}_j} \|(I - K_f)^{-1}\| < \infty.$$

Thus the integral equation (5.6) and its adjoint equation are uniquely solvable in $[L^p(\mathbb{R})]^2$ for any righthand side $\mathbf{g} \in [L^p(\mathbb{R})]^2$ and the solution depends continuously on the righthand side.

Chapter 6

Concluding Remarks

The results presented in this thesis form a detailed investigation into the questions of uniqueness and existence of solution for the elastic wave scattering problem for a rough surface on which the displacement vanishes. It has been shown that for a general class of incident fields, including the special cases of cylindrical and plane waves, a unique solution exists. To obtain the uniqueness result, a novel radiation condition has been proposed characterising upward propagating wave fields, and it has been shown that this condition generalises Kupradze's radiating condition as well as the Rayleigh expansion radiation condition commonly used in diffraction grating problems.

To prove existence of solution, the regularity of elastic single- and double-layer potentials defined on bounded obstacles and on rough surfaces has been studied in detail. The estimates obtained differ from conventional results in that they hold uniformly for all surfaces sharing certain geometric properties. The rough surface potentials have then been used in a combined single- and double-layer potential *ansatz* for the scattering field. A novel solvability theory has been employed to prove solvability of the resulting boundary integral equation in the space of bounded and continuous functions as well as in the all L^p-spaces, $1 \leq p \leq \infty$.

The results presented appear complete in the sense that they give an affirmative answer to the question of unique solvability for the elastic wave scattering problem for a rigid rough surface. However, a number of remarks are in order on, as of yet, not fully satisfactory results as well as on important consequences.

A slighly limiting aspect of the presented results is the restriction to surfaces of class $C^{1,1}$ as opposed to those of class $C^{1,\alpha}$ in Chapter 5. In fact, the solvability theory itself does not rely on this assumption; it is only due to the nature of the regularity results of Chapter 3 and the corresponding mapping properties of the integral operators. An extension of these results to boundaries of class $C^{1,\alpha}$, $\alpha \in (0, 1]$, appears possible but the amount of technical detail involved seems disproportionate to that of new insights being gained. With regard to even more general surfaces, it seems possible to extend the uniqueness results to scattering surfaces that are piecewise Lyapunov; for the acoustic wave case this result has been proven in [20] and it is anticipated that similar arguments apply for the elastic wave case. However, the question of solvability of boundary integral equations for such surfaces remains open.

A further issue of interest is the extension of the results presented in Chapters 4 and 5 to scattering problems involving obstacles with different physical properties. From the point of view of applications, a free surface or a Robin boundary condition would be more realistic than the rigid surface considered here. As regards the question of uniqueness of solution, very similar methods to those used in Chapter 4 could be applied in the case of a Robin boundary condition; it is even anticipated that the arguments simplify to some extend in this case. On the other hand, uniqueness of solution does not hold in the free surface case if the same general class of incident fields is permitted as in the present investigation. This is due to the fact that Problem 4.15 with the Dirichlet boundary condition replaced by $\mathbf{P}\mathbf{u} = \mathbf{g}$ admits Rayleigh surface wave solutions in the case $\mathbf{g} = 0$.

To prove existence of solution to either problem by the boundary integral equation approach, requires, for any physically realistic problem, the use of the stress tensor σ_{jk} instead of the generalised stress tensor π_{jk} and thus prohibits the use of the pseudo stress operator. As a consequence, it is necessary to deal with integral operators of Cauchy singular type. It is an open question of some interest, if and how the solvability theory developed in Chapter 5 can be extended to this case.

On the other hand, the presented approach to prove solvability of the boundary integral operators in all L^p-spaces can be applied directly to a wide class of integral operators arising in acoustic and electro-magnetic scattering theory. A first such application is presented in [6] for the case of acoustic waves.

An issue that has not been the subject of this investigation is the numerical computation of the solution to a rough surface scattering problem. With the use of the boundary integral equation method this reduces to the numerical solution of a boundary integral equation, or systems of such equations, on the real line. The principal method that has been proposed for this purpose is the *finite section method* which amounts to limiting the range of integration to a finite interval $[-A, A]$, and then proving convergence of the corresponding finite section solution as $A \to \infty$ [39, 42]. The integral equation with the truncated range of integration can be solved in principal by any collocation, quadrature or Galerkin method of choice, but the resulting dense linear system needs to be large for accuracy and thus sophisticated matrix compression schemes need to be employed for the solution. Examples of such methods are described in [6, 13]

Appendix A

Regularity Results up to the Boundary for Second Order Elliptic Systems

In this appendix, regularity results obtained for scalar elliptic equations of the second order in Gilbarg/Trudinger [28] are generalised for systems of such equations. It is the objective to show that the weak solution of any elliptic equation with sufficiently regular coefficients in a bounded $C^{1,\alpha}$-domain is $C^{1,\alpha}$ up to the boundary provided its boundary values are in that space.

After some definitions and preliminary inequalities in Section 1, $C^{1,\alpha}$-estimates, both interior and *up to the boundary*, are proved for weak solutions to elliptic systems under suitable assumptions on the smoothness of the coefficient functions, the domain and the boundary values in Section 2. These estimates are used in the third section to prove existence of weak $C^{1,\alpha}$-solutions and, subsequently, regularity of ordinary weak solutions up to the boundary.

A.1 Definitions and Interpolation Inequalities

The systems of partial differential equations under consideration are of the form

$$L_k^l u_l = g_k + \operatorname{div} \mathbf{f_k}, \qquad k = 1, \ldots, m \tag{A.1}$$

in some bounded domain $D \subset \mathbb{R}^n$. Here and in much of the following, we make use of the usual summation convention, i.e. a sum will be taken over all repeated indices.

The linear differential operators L_k^l are assumed to be of the form

$$L_k^l v := a_k^{lij}(\mathbf{x})D_{ij}v + b_k^{li}(\mathbf{x})D_i v + c_k^l(\mathbf{x})v, \qquad k,l = 1,\dots,m,$$

where we will assume throughout that the coefficient functions satisfy $a_k^{lij} \in C^{1,\alpha}(\bar{D})$, $b_k^{li} \in C^\alpha(\bar{D})$, $c_k^l \in C(\bar{D})$ and that there exist constants λ, $\Lambda > 0$ with

$$a_k^{lij}(\mathbf{x})\xi_i\xi_j \geq \lambda|\xi|^2$$

for all $\mathbf{x} \in D$, $\xi \in \mathbb{R}^n$, $k,l = 1,\dots,m$ and

$$\|a_k^{lij}\|_{1,\alpha;D},\ \|b_k^{li}\|_{0,\alpha;D},\ \|c_k^l\|_{\infty;D} \leq \Lambda$$

for all $k,l = 1,\dots,m$, $i,j = 1,\dots n$. Unless specifically stated otherwise for some individual results, we will also assume $\mathbf{g} \in [C(\bar{D})]^m$ and $\mathbf{f_k} \in [C^\alpha(\bar{D})]^m$, $(k = 1,\dots,m)$.

Definition A.1 *A vector field $\mathbf{u} \in [H^1(D)]^m$ is called a weak solution to the system (A.1), if, for any test function $\mathbf{v} \in [H_0^1(D)]^m$, the equation*

$$\int_D \left(a_k^{lij}\, D_i u_l\, D_j v_k + (D_j a_k^{lij} - b_k^{li})\, D_i u_l\, v_k - c_k^l\, u_l\, v_k\right) d\mathbf{x}$$
$$= -\int_D \left(g_k\, v_k + f_{k,i}\, D^i v_k\right) d\mathbf{x}$$

holds.

For $\mathbf{x},\mathbf{y} \in D$, set $d_\mathbf{x} := \operatorname{dist}(\mathbf{x},\partial D)$ and $d_{\mathbf{x},\mathbf{y}} := \min\{d_\mathbf{x}, d_\mathbf{y}\}$. For $\sigma \in \mathbb{N}_0$ and $u \in C^k(D)$ we define

$$\begin{aligned} [u]_{k;D}^{(\sigma)} &:= \sup_{\substack{\mathbf{x}\in D\\|\beta|=k}} d_\mathbf{x}^{k+\sigma}|D^\beta u(\mathbf{x})|,\\ |u|_{k;D}^{(\sigma)} &:= \sum_{j=0}^k [u]_{j,D}^{(\sigma)}, \end{aligned}$$

and for $u \in \mathcal{V}^{k,\alpha}(D)$,

$$\begin{aligned} [u]_{k,\alpha;D}^{(\sigma)} &:= \sup_{\substack{\mathbf{x},\mathbf{y}\in D\\|\beta|=k}} d_{\mathbf{x},\mathbf{y}}^{k+\alpha+\sigma}\frac{|D^\beta u(\mathbf{x}) - D^\beta u(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\alpha},\\ |u|_{k,\alpha;D}^{(\sigma)} &:= |u|_{k;D}^{(\sigma)} + [u]_{k,\alpha;D}^{(\sigma)}. \end{aligned}$$

Assuming T to be a (possibly empty) portion of ∂D, we set $\bar{d}_\mathbf{x} := \operatorname{dist}(\mathbf{x},\partial D \setminus T)$ and $\bar{d}_{\mathbf{x},\mathbf{y}} := \min\{\bar{d}_\mathbf{x}, \bar{d}_\mathbf{y}\}$. Using these notations, we define the semi-norms $[u]_{k;D\cup T}^{(\sigma)}$, $[u]_{k,\alpha;D\cup T}^{(\sigma)}$, $|u|_{k;D\cup T}^{(\sigma)}$ and $|u|_{k,\alpha;D\cup T}^{(\sigma)}$ as above, only replacing $d_\mathbf{x}$ by $\bar{d}_\mathbf{x}$ and $d_{\mathbf{x},\mathbf{y}}$ by $\bar{d}_{\mathbf{x},\mathbf{y}}$ respectively. All these definitions are easily extended to vector fields by taking the sum of their components. We then have the following simple extensions of some results in [28]:

Lemma A.2 (p. 61 in [28]) *Assume $D \subset \mathbb{R}^2$ to be bounded with $d := \mathrm{diam}(D)$ and $\mathbf{u} \in [\mathcal{V}^{k,\alpha}(D)]^m$. Then,*

(a) $|\mathbf{u}|^{(0)}_{k,\alpha;D} \leq \max\{1, d^{k+\alpha}\} \sup_{D' \subset\subset D} \|\mathbf{u}\|_{k,\alpha;D'}$,

(b) for any $D' \subset\subset D$ and $s := \mathrm{dist}\,(D', \partial D)$, there holds

$$\min\{1, s^{k+\alpha}\}\, \|\mathbf{u}\|_{k,\alpha;D'} \leq |\mathbf{u}|^{(0)}_{k,\alpha;D}.$$

Lemma A.3 (Lemma 6.32 of [28]) *Suppose $j + \beta < k + \alpha$, where $j, k = 0, 1$ and $0 \leq \alpha, \beta \leq 1$. Let $\mathbf{u} \in [\mathcal{V}^{k,\alpha}(D)]^m$. Then for any $\varepsilon > 0$ and some constant $C = C(\varepsilon, k, j, n, m)$ we have*

$$\begin{aligned} [\mathbf{u}]^{(0)}_{j,\beta;D} &\leq C\,\|\mathbf{u}\|_{\infty;D} + \varepsilon[\mathbf{u}]^{(0)}_{k,\alpha;D}, \\ |\mathbf{u}|^{(0)}_{j,\beta;D} &\leq C\,\|\mathbf{u}\|_{\infty;D} + \varepsilon[\mathbf{u}]^{(0)}_{k,\alpha;D}. \end{aligned}$$

A.2 Estimates for Weak Solutions

From results for scalar elliptic equations, it is straight-forward to obtain the following result for a special, very simple case of the system (A.1). Assume that there exist constants A^{ij}_k $(i, j = 1, \ldots, n,\ k = 1, \ldots, m)$ with

$$a^{lij}_k = \delta_{lk}\, A^{ij}_k, \quad \text{and that} \quad b^{li}_k = 0,\ c^l_k = 0. \tag{A.2}$$

Then we have the following lemma, which is proved similarly as Lemma 6.1 of [28], but is based on estimates (4.45) and (4.46) in that same reference:

Lemma A.4 *Assume (A.2) holds and* $\mathbf{u}$ *is a bounded, weak solution to (A.1) in D with $\mathbf{u} \in [\mathcal{V}^{1,\alpha}(D)]^m$. Then*

$$|u_k|^{(0)}_{1,\alpha;D} \leq C(\|u_k\|_{\infty;D} + |g_k|^{(2)}_{0;D} + |\mathbf{f_k}|^{(1)}_{0,\alpha;D}), \qquad k = 1, \ldots, m,$$

where C is a constant only depending on n, m, α, λ and Λ. If we further assume $D \subset \mathbb{R}^n_+$, $T \subset \{x_n = 0\}$ to be a boundary portion of D, $\mathbf{u} \in [C^{1,\alpha}(D \cup T)]^m$ and $\mathbf{u} = 0$ on T, then

$$|u_k|^{(0)}_{1,\alpha;D\cup T} \leq C(\|u_k\|_{\infty;D} + |g_k|^{(2)}_{0;D\cup T} + |\mathbf{f_k}|^{(1)}_{0,\alpha;D\cup T}), \qquad k = 1, \ldots, m,$$

where C is a constant only depending on n, m, α, λ and Λ.

We will now derive inequalities for weak solutions to (A.1) under general assumptions. The first aim will be to derive interior estimates, given in the following lemma:

Lemma A.5 *Let* $\mathbf{u}$ *be a bounded, weak solution to (A.1) in* D *with* $\mathbf{u} \in [\mathcal{V}^{1,\alpha}(D)]^m$. *Then*

$$|\mathbf{u}|^{(0)}_{1,\alpha;D} \leq C \left(\|\mathbf{u}\|_{\infty;D} + |\mathbf{g}|^{(2)}_{0;D} + \sum_{j=1}^{k} |\mathbf{f_k}|^{(1)}_{0,\alpha;D} \right)$$

where C *depends only on* n, m, α, λ *and* Λ.

Proof: The method of proof follows very closely the argument in the proof of Theorem 6.2 in [28]. Analogously to the first argument given there, it suffices to prove the asserted estimate for $[\mathbf{u}]^{(0)}_{1,\alpha;D}$ and we may assume $[\mathbf{u}]^{(0)}_{1,\alpha;D}$ to be finite.

Choose $\mathbf{x}_0$ and $\mathbf{y}_0 \in D$ arbitrarily but assume without loss of generality that $d_{\mathbf{x}_0} \leq d_{\mathbf{y}_0}$, so that $d_{\mathbf{x}_0} = d_{\mathbf{x}_0,\mathbf{y}_0}$. Further let $\mu \leq \frac{1}{2}$ be a positive constant (which will be specified later) and set $d := \mu d_{\mathbf{x}_0}$, $B := B_d(\mathbf{x}_0)$.

For convenience of notation we introduce

$$M^{(i)}_k \mathbf{u} := \begin{cases} b^{li}_k(\mathbf{x}) D_i u_l - \left(D_j a^{lji}_k(\mathbf{x}) \right) D_i u_l, & i = 0, \\ \\ \sum_{\substack{l=1 \\ l \neq k}}^{m} a^{lij}_k(\mathbf{x}) D_j u_l, & i = 1, \ldots, n, \end{cases}$$

and now rewrite (A.1) as

$$a^{kij}_k(\mathbf{x}_0) D_{ij} u_k = G_k + \operatorname{div} \mathbf{F_k} \tag{A.3}$$

where

$$\begin{aligned} G_k &:= -M^{(0)}_k \mathbf{u} - c^l_k(\mathbf{x}) u_l + g_k, \\ F_{k,i} &:= \left\{ \left(a^{kij}_k(\mathbf{x}_0) - a^{kij}_k(\mathbf{x}) \right) D_j u_k \right\} - M^{(i)}_k \mathbf{u} + f_{k,i}. \end{aligned}$$

We can now apply Lemma A.4 to equation (A.3). Let $\mathbf{y}_0 \in B_{d/2}(\mathbf{x}_0)$. Then,

$$\left(\frac{d}{2} \right)^{1+\alpha} \frac{|D^1 u_k(\mathbf{x}_0) - D^1 u_k(\mathbf{y}_0)|}{|\mathbf{x}_0 - \mathbf{y}_0|^\alpha} \leq \frac{C}{\mu^{1+\alpha}} \left(\|u_k\|_{\infty;B} + |G_k|^{(2)}_{0;B} + |\mathbf{F_k}|^{(1)}_{0,\alpha;B} \right),$$

where $D^1 u_k$ denotes any first derivative of u_k, and thus

$$d^{1+\alpha}_{\mathbf{x}_0} \frac{|D^1 u_k(\mathbf{x}_0) - D^1 u_k(\mathbf{y}_0)|}{|\mathbf{x}_0 - \mathbf{y}_0|^\alpha} \leq \frac{C}{\mu^{1+\alpha}} \left(\|u_k\|_{\infty;B} + |G_k|^{(2)}_{0;B} + |\mathbf{F_k}|^{(1)}_{0,\alpha;B} \right).$$

If, on the other hand, we have $|\mathbf{x}_0 - \mathbf{y}_0| \geq d/2$, then it is easy to see that

$$\begin{aligned} d_{\mathbf{x}_0}^{1+\alpha} \frac{|D^1 u_k(\mathbf{x}_0) - D^1 u_k(\mathbf{y}_0)|}{|\mathbf{x}_0 - \mathbf{y}_0|^\alpha} &\leq \left(\frac{2}{\mu}\right)^\alpha \left(d_{\mathbf{x}_0}|D^1 u_k(\mathbf{x_0})| + d_{\mathbf{y}_0}|D^1 u_k(\mathbf{y}_0)|\right) \\ &\leq \frac{4}{\mu^\alpha}[u_k]^{(0)}_{1;D}. \end{aligned}$$

Combining these two estimates yields

$$\begin{aligned} & d_{\mathbf{x}_0}^{1+\alpha} \frac{|D^1 u_k(\mathbf{x}_0) - D^1 u_k(\mathbf{y}_0)|}{|\mathbf{x}_0 - \mathbf{y}_0|^\alpha} \\ & \qquad \leq \frac{C}{\mu^{1+\alpha}} \left(\|u_k\|_{\infty;D} + |G_k|^{(2)}_{0;B} + |\mathbf{F_k}|^{(1)}_{0,\alpha;B}\right) + \frac{4}{\mu^\alpha}[u_k]^{(0)}_{1;D}. \end{aligned} \tag{A.4}$$

Now, for $\mathbf{x} \in B$ there holds $d_\mathbf{x} > (1-\mu)\, d_{\mathbf{x}_0} \geq \frac{1}{2} d_{\mathbf{x}_0}$. Thus, for any $h \in C^\alpha(D)$, we have analogously to equation (6.18) in [28]

$$\begin{aligned} |h|^{(1)}_{0,\alpha;B} &\leq d\|h\|_{\infty;B} + d^{1+\alpha}[h]_{\alpha;B} \\ &\leq \frac{\mu}{1-\mu}|h|^{(1)}_{0;D} + \frac{\mu^{1+\alpha}}{(1-\mu)^{1+\alpha}}[h]^{(1)}_{0,\alpha;D} \\ &\leq 2\mu|h|^{(1)}_{0;D} + 4\mu^{1+\alpha}[h]^{(1)}_{0,\alpha;D} && \text{(A.5)} \\ &\leq 4\mu|h|^{(1)}_{0,\alpha;D}. && \text{(A.6)} \end{aligned}$$

These two inequalities will now be used to further estimate (A.4). Firstly, we have

$$|F_{k,i}|^{(1)}_{0,\alpha;B} \leq \sum_{j=1}^n \left|(a_k^{kij}(\mathbf{x}_0) - a_k^{kij}(\mathbf{x}))D_j u_k\right|^{(1)}_{0,\alpha;B} + |M_k^i \mathbf{u}|^{(1)}_{0,\alpha;B} + |f_{k,i}|^{(1)}_{0,\alpha;B}.$$

Using (A.6) yields

$$|f_{k,i}|^{(1)}_{0,\alpha;B} \leq 4\mu|f_{k,i}|^{(1)}_{0,\alpha;D}$$

and

$$\begin{aligned} |M_k^i \mathbf{u}|^{(1)}_{0,\alpha;B} &\leq \sum_{l=1}^m \sum_{j=1}^n |a_k^{lji} D_i u_l|^{(1)}_{0,\alpha;B} \\ &\leq 4\mu \max_{l,j} |a_k^{lji} D_i u_l|^{(1)}_{0,\alpha;D} \\ &\leq 4\mu\Lambda|\mathbf{u}|^{(0)}_{1,\alpha;D} \\ &\leq 4\mu\Lambda\left(C(\mu)\|\mathbf{u}\|_{\infty;D} + \mu^{2\alpha}[\mathbf{u}]^{(0)}_{1,\alpha;D}\right), \end{aligned}$$

where the last inequality is obtained by applying Lemma A.3 with $\varepsilon = \mu^{2\alpha}$. Using (A.5) and $|a_k^{kij}(\mathbf{x}_0) - a_k^{kij}(\cdot)|^{(0)}_{0,\alpha;B} \leq 4\Lambda\mu^\alpha$ and again Lemma A.3, we also obtain the estimate

$$\sum_{j=1}^n \left|(a_k^{kij}(\mathbf{x}_0) - a_k^{kij}(\cdot))D_j u_k\right|^{(1)}_{0,\alpha;B} \leq 16n\Lambda\mu^{1+\alpha}\left(C(\mu)\|u_k\|_{\infty;D} + 2\mu^\alpha[u_k]^{(0)}_{1,\alpha;D}\right).$$

Combining these last three inequalites yields the bound

$$|\mathbf{F}_{\mathbf{k}}|^{(1)}_{0,\alpha;B} \leq C\mu^{1+2\alpha}[\mathbf{u}]^{(0)}_{1,\alpha;D} + C(\mu)\left(\|\mathbf{u}\|_{\infty;D} + |\mathbf{f}_{\mathbf{k}}|^{(1)}_{0,\alpha;D}\right). \tag{A.7}$$

In a similar fashion, we conclude

$$|G_k|^{(2)}_{0;B} \leq C\mu^{1+2\alpha}[\mathbf{u}]^{(0)}_{1,\alpha;D} + C(\mu)\left(\|\mathbf{u}\|_{\infty;D} + |g_k|^{(2)}_{0,\alpha;D}\right). \tag{A.8}$$

Applying Lemma A.3 one final time, we estimate

$$\frac{4}{\mu^\alpha}[u_k]^{(0)}_{1;D} \leq 4\left(C(\mu)\|u_k\|_{\infty;D} + \mu^\alpha[u_k]^{(0)}_{1,\alpha;D}\right).$$

Combining this result with (A.4), (A.7) and (A.8) now yields

$$[u_k]^{(0)}_{1,\alpha;D} \leq C\mu^\alpha[\mathbf{u}]^{(0)}_{1,\alpha;D} + C(\mu)\left(\|\mathbf{u}\|_{\infty;D} + |g_k|^{(2)}_{0,\alpha;D} + |\mathbf{f}_{\mathbf{k}}|^{(1)}_{0,\alpha;D}\right).$$

Summing up this last estimate over $k = 1, \ldots, m$ and choosing μ so that $Cm\mu^\alpha \leq \frac{1}{2}$, we now finally obtain

$$[\mathbf{u}]^{(0)}_{1,\alpha;D} \leq C\left(\|\mathbf{u}\|_{\infty;D} + |\mathbf{g}|^{(2)}_{0;D} + \sum_{j=1}^{k}|\mathbf{f}_{\mathbf{k}}|^{(1)}_{0,\alpha;D}\right).$$

■

In the presence of a flat boundary portion, this result is easily extended to yield an estimate up to the boundary:

Lemma A.6 *Let $D \subseteq \mathbb{R}^n_+$, $T \subseteq \{x_n = 0\}$ a boundary portion of D, $\mathbf{u}$ a bounded, weak solution to equation (A.1) with $\mathbf{u} \in [\mathcal{V}^{1,\alpha}(D \cup T)]^m$ and $\mathbf{u} = 0$ on T. Then there holds*

$$|\mathbf{u}|^{(0)}_{1,\alpha;D\cup T} \leq C\left(\|\mathbf{u}\|_{\infty;D} + |\mathbf{g}|^{(2)}_{0;D\cup T} + \sum_{j=1}^{k}|\mathbf{f}_{\mathbf{k}}|^{(1)}_{0,\alpha;D\cup T}\right).$$

where C depends only on n, m, α, λ and Λ.

Proof: The proof is essentially identical to that of Lemma A.5; it is only necessary to replace $d_{\mathbf{x}}$ by $\bar{d}_{\mathbf{x}}$. ■

We will now generalise Lemma A.6 to obtain estimates up to the boundary for weak solutions to elliptic systems in domains with sufficiently smooth boundaries. For technical simplicity, we will limit ourselves to domains in $\mathbb{R}^2$, but note that all arguments carry through for domains in $\mathbb{R}^n$, $n > 2$. Recalling the definition of the set of $C^{1,\alpha}$ domains, $\mathcal{D}_{\alpha,\kappa_0,\delta,M}$, from Section 1.3, we now consider, for κ_0, δ, $M > 0$, domains $D \in \mathcal{D}_{\alpha,\kappa_0,\delta,M}$.

We can obtain uniform estimates up to the boundary for weak solutions to elliptic systems in such domains with vanishing boundary values:

Lemma A.7 *Let $D \in \mathcal{D}_{\alpha,\kappa_0,\delta,M}$ and $\mathbf{u} \in C^{1,\alpha}(\bar{D})$ a weak solution to (A.1) in D with $\mathbf{u} = 0$ on ∂D. Then there exists $\varepsilon > 0$ so that for each $\mathbf{x_0} \in \partial D$, setting $B := B_\varepsilon(\mathbf{x}_0)$, there holds*

$$\|\mathbf{u}\|_{1,\alpha;B\cap D} \leq C \left(\|\mathbf{u}\|_{\infty;D} + \|\mathbf{g}\|_{\infty;D} + \sum_{k=1}^{m} \|\mathbf{f_k}\|_{0,\alpha;D} \right),$$

where the constant C depends only on m, α, λ, Λ, κ_0, δ and M.

Proof: Using the estimates for $C^{k,\alpha}$-diffeomorphisms on page 96 of [28] and applying the same arguments as in the proof of Lemma 6.5 in [28], Lemma A.6 implies that for each $\mathbf{x_0} \in \partial D$ there exists a ball $B_\rho(\mathbf{x}_0)$ so that

$$|\mathbf{u}|^{(0)}_{1,\alpha;B'\cup T} \leq C\, K(\mathbf{x_0}) \left(\|\mathbf{u}\|_{\infty;D} + |\mathbf{g}|^{(2)}_{0;B'\cup T} + \sum_{j=1}^{k} |\mathbf{f_k}|^{(1)}_{0,\alpha;B'\cup T} \right),$$

where $B' := B_\rho(\mathbf{x}_0) \cap D$ and $T := B_\rho(\mathbf{x}_0) \cap \partial D$ and $K(\mathbf{x_0})$ is a constant depending only on the $C^{1,\alpha}$ diffeomorphism corresponding to $\mathbf{x_0}$. Thus, as $D \in \mathcal{D}_{\alpha,\kappa_0,\delta,M}$, it follows that K can in fact be chosen independently of $\mathbf{x}_0$ and dependent only on α, κ_0, δ and M. Now set $B'' := B_{\rho/2}(\mathbf{x}_0) \cap D$. Then Lemma A.2 yields

$$\min\left\{1, (\rho/2)^{1+\alpha}\right\} \|\mathbf{u}\|_{1,\alpha;B''} \leq |\mathbf{u}|^{(0)}_{1,\alpha;B'\cup T}.$$

Thus we have the result

$$\|\mathbf{u}\|_{1,\alpha;B''} \leq C\, K \left(\|\mathbf{u}\|_{\infty;D} + \|\mathbf{g}\|_{\infty;D} + \sum_{k=1}^{m} \|\mathbf{f_k}\|_{0,\alpha;D} \right). \tag{A.9}$$

As the set of balls $\{B_{\rho(\mathbf{x})/4}(\mathbf{x}) : \mathbf{x} \in \partial D\}$ covers all of ∂D, the compactness of ∂D implies existence of a finite set $\{B_{\rho_i/4}(\mathbf{x}_i) : i = 1, \ldots, N\}$ that covers ∂D. Set $\varepsilon := \min_{i=1,\ldots,N} \{\rho_i/4\}$ and $\mathcal{X} := \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$. Now, choosing $\mathbf{x} \in \partial D$ arbitrarily and setting $B := B_\varepsilon(\mathbf{x})$ we conclude $B \subseteq B_{\rho/2}(\mathbf{x}_0)$ for some $\mathbf{x}_0 \in \mathcal{X}$. Thus, by (A.9),

$$\begin{aligned} \|\mathbf{u}\|_{1,\alpha;B\cap D} &\leq \|\mathbf{u}\|_{1,\alpha;B_{\rho/2}(\mathbf{x}_0)\cap D} \\ &\leq C\, K \left(\|\mathbf{u}\|_{\infty;D} + \|\mathbf{g}\|_{\infty;D} + \sum_{k=1}^{m} \|\mathbf{f_k}\|_{0,\alpha;D} \right). \end{aligned}$$

■

As a consequence of the results obtained so far, it is now possible to formulate global bounds up to the boundary for solutions to elliptic systems in $C^{1,\alpha}$-domains:

Theorem A.8 *Assume $D \in \mathcal{D}_{\alpha,\kappa_0,\delta,M}$ and $\mathbf{u} \in [C^{1,\alpha}(\bar{D})]^m$ to be a weak solution to (A.1) satisfying $\mathbf{u} = \phi$ on ∂D for some $\phi \in [C^{1,\alpha}(\bar{D})]^m$. Then*

$$\|\mathbf{u}\|_{1,\alpha;D} \leq C \left(\|\mathbf{u}\|_{\infty;D} + \|\phi\|_{1,\alpha;D} + \|\mathbf{g}\|_{\infty;D} + \sum_{k=1}^{m} \|\mathbf{f_k}\|_{0,\alpha;D} \right),$$

where the constant C depends only on m, α, λ, Λ, κ_0, δ ,M.

Proof: Following the argument of the proof of Theorem 6.6 in [28], it suffices to prove the assertion for $\phi \equiv 0$. Assume $\mathbf{x} \in D$ and let ε denote the radius in Lemma A.7. Further let $D' \subset\subset D$ satisfy $\operatorname{dist}(D', \partial D) > \varepsilon/2$.

If $\mathbf{x} \in B_\varepsilon(\mathbf{x}_0) \cap D$ for some $\mathbf{x}_0 \in \partial D$, then Lemma A.7 implies

$$|D^1\mathbf{u}(\mathbf{x})| \leq C_1' \left(\|\mathbf{u}\|_{\infty;D} + \|\mathbf{g}\|_{\infty;D} + \sum_{k=1}^{m} \|\mathbf{f_k}\|_{0,\alpha;D} \right)$$

for any first derivative $D^1\mathbf{u}$ of $\mathbf{u}$.

On the other hand, if $\mathbf{x} \in D'$, Lemma A.5 implies

$$d_{\mathbf{x}}|D^1\mathbf{u}(\mathbf{x})| \leq C_1'' \left(\|\mathbf{u}\|_{\infty;D} + \|\mathbf{g}\|_{\infty;D} + \sum_{k=1}^{m} \|\mathbf{f_k}\|_{0,\alpha;D} \right).$$

As $d_{\mathbf{x}} \geq \varepsilon/2$, combining these two estimates yields

$$|D^1\mathbf{u}(\mathbf{x})| \leq C_1 \left(\|\mathbf{u}\|_{\infty;D} + \|\mathbf{g}\|_{\infty;D} + \sum_{k=1}^{m} |\mathbf{f_k}|_{0,\alpha;D} \right) \tag{A.10}$$

for all $\mathbf{x} \in D$.

Now choose $\mathbf{x}$, $\mathbf{y} \in D$ arbitrarily. Three cases have to be considered:

1. $\mathbf{x}, \mathbf{y} \in B_\varepsilon(\mathbf{x}_0) \cap D$ for some $\mathbf{x}_0 \in \partial D$,
2. $\mathbf{x}, \mathbf{y} \in D'$,

3. $\mathbf{x}$ or $\mathbf{y} \in D \setminus D'$, but not in the same ball $B_\varepsilon(\mathbf{x}_0)$ for any $\mathbf{x}_0$.

In the first two cases we proceed as above employing Lemma A.7 and Lemma A.5 respectively to obtain the required bounds on $\frac{|D^1\mathbf{u}(\mathbf{x})-D^1\mathbf{u}(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\alpha}$. In the third case, dist $(\mathbf{x},\mathbf{y}) > \varepsilon/2$ and therefore

$$\begin{aligned}\frac{|D^1\mathbf{u}(\mathbf{x}) - D^1\mathbf{u}(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\alpha} &\leq \left(\frac{\varepsilon}{2}\right)^{-\alpha}\left(|D^1\mathbf{u}(\mathbf{x}) - D^1\mathbf{u}(\mathbf{y})|\right) \\ &\leq \left(\frac{\varepsilon}{2}\right)^{-\alpha}\left(|D^1\mathbf{u}(\mathbf{x})| + |D^1\mathbf{u}(\mathbf{y})|\right) \\ &\leq C_3\left(\|\mathbf{u}\|_{\infty;D} + \|\mathbf{g}\|_{\infty;D} + \sum_{k=1}^m \|\mathbf{f_k}\|_{0,\alpha;D}\right)\end{aligned}$$

by (A.10).

Combining these last estimates with (A.10) yields the assertion. ∎

A similar estimate can also be obtained for a domain that only has a boundary portion of class $C^{1,\alpha}$. We introduce the set $\mathcal{T}_{\alpha,\kappa_0,\delta,M}$ of domains D such that there exists a boundary portion $T \subset \partial D$ with the property that for every $T' \subset\subset T$ there exists $D' \in \mathcal{D}_{\alpha,\kappa_0,\delta,M}$ such that $D' \subset D$ and $T' \subset \partial D' \cap \partial D$. For $D \in \mathcal{T}_{\alpha,\kappa_0,\delta,M}$, the set T in this definition is called a $C^{1,\alpha}$ boundary portion of ∂D.

Corollary A.9 *Let $D \in \mathcal{T}_{\alpha,\kappa_0,\delta,M}$ and T a $C^{1,\alpha}$-boundary portion of ∂D. Further assume $\mathbf{u} \in [\mathcal{V}^{1,\alpha}(D \cup T)]^m$ to be a weak solution to (A.1) and $\mathbf{u} = \phi$ on T where $\phi \in [C^{1,\alpha}(\bar{D})]^m$. Then, for every $\mathbf{x}_0 \in T$ and $0 < \rho <$ dist$(\mathbf{x}_0, \partial D \setminus T)$, there holds*

$$\|\mathbf{u}\|_{1,\alpha;B_\rho(\mathbf{x}_0)\cap D} \leq C\left(\|\mathbf{u}\|_{\infty;D} + \|\phi\|_{1,\alpha;D} + \|\mathbf{g}\|_{\infty;D} + \sum_{k=1}^m \|\mathbf{f_k}\|_{0,\alpha;D}\right),$$

where the constant C depends only on n, m, α, λ, Λ, δ, κ_0, M and ρ.

Proof: Set $T' := \{\mathbf{x} \in T : |\mathbf{x} - \mathbf{x}_0| < \rho\}$ and choose $D' \in \mathcal{D}_{\alpha,\kappa_0,\delta,M}$ such that $D' \subset D$ and $T' \subset \partial D' \cap \partial D$. Now apply Theorem A.8 in D' and employ an interior estimate on $(B_\rho(\mathbf{x}_0) \cap D) \setminus D'$, if necessary. ∎

A.3 Regularity of Weak Solutions

To later obtain the regularity results for weak solutions to elliptic systems, we first need to prove the existence of a regular solution in the case where uniqueness is already ensured:

Theorem A.10 *Let $D \subset \mathbb{R}^2$ be a bounded domain of class $C^{1,\alpha}$ so that the boundary value problem*

$$\begin{aligned} a_k^{lij}(\mathbf{x})D_{ij}u_l + b_k^{li}(\mathbf{x})D_i u_l + c_k^l(\mathbf{x})u_l = 0 \qquad & \text{in } D, \quad k = 1,\ldots,m \\ u = 0 \qquad & \text{on } \partial D \end{aligned}$$

only has the trivial solution in $[H_0^1(D)]^m$. Then, for $\mathbf{g} \in [L^\infty(D)]^m$, $\mathbf{f_k} \in [C^\alpha(\bar{D})]^m$ ($k = 1,\ldots,m$) and $\phi \in [C^{1,\alpha}(\bar{D})]^m$, the boundary value problem

$$\begin{aligned} a_k^{lij}(\mathbf{x})D_{ij}u_l + b_k^{li}(\mathbf{x})D_i u_l + c_k^l(\mathbf{x})u_l = g_k + \operatorname{div}\mathbf{f_k} \qquad & \text{in } D, \quad k = 1,\ldots,m, \\ u = \phi \qquad & \text{on } \partial D \end{aligned} \tag{A.11}$$

has a uniquely determined weak solution $\mathbf{u} \in [C^{1,\alpha}(\bar{D})]^m$.

Proof: Let Γ be the set of coefficient functions in (A.11). Approximate any $\gamma \in \Gamma$ uniformly in D by a sequence (γ_ν) in $C^\infty(\bar{D})$. Similarly, choose $[C^\infty(\bar{D})]^m$ sequences $(\mathbf{g}_\nu)$, $(\mathbf{f}_{\mathbf{k}\nu})$ and (ϕ_ν) that uniformly converge to $\mathbf{g}$ in $[L^\infty(D)]^m$, $\mathbf{f_k}$ in $[C^{0,\alpha}(\bar{D})]^m$ and ϕ in $[C^{1,\alpha}(\bar{D})]^m$, respectively. Finally, let (D_ν) be a sequence of C^∞-domains that exhausts D from the inside and whose members are uniformly of class $C^{1,\alpha}$.

From results in Fichera [26] it follows that the approximations to the boundary value problem (A.11) have unique solutions $\mathbf{u}_\nu \in [C^\infty(\bar{D}_\nu)]^m$. An application of Theorem A.8 yields the estimate

$$\|\mathbf{u}_\nu\|_{1,\alpha;D_\nu} \le C\left(\|\mathbf{u}_\nu\|_{\infty;D_\nu} + \|\phi_\nu\|_{1,\alpha;D_\nu} + \|\mathbf{g}_\nu\|_{\infty;D_\nu} + \sum_{k=1}^m \|\mathbf{f}_{\mathbf{k}\nu}\|_{0,\alpha;D_\nu}\right). \tag{A.12}$$

From the maximum principle we also can estimate

$$\|\mathbf{u}_\nu\|_{\infty;D_\nu} \le \|\phi_\nu\|_{\infty;D_\nu} + C\left(\|\mathbf{g}_\nu\|_{\infty;D_\nu} + \sum_{k=1}^m \|\mathbf{f}_{\mathbf{k}\nu}\|_{\infty;D_\nu}\right).$$

Combining this with (A.12) yields an estimate of the form

$$\begin{aligned} \|\mathbf{u}_\nu\|_{1,\alpha;D_\nu} &\le C\left(\|\phi_\nu\|_{1,\alpha;D_\nu} + \|\mathbf{g}_\nu\|_{\infty;D_\nu} + \sum_{k=1}^m \|\mathbf{f}_{\mathbf{k}\nu}\|_{0,\alpha;D_\nu}\right) \\ &\le C\left(\|\phi\|_{1,\alpha;D} + \|\mathbf{g}\|_{\infty;D} + \sum_{k=1}^m \|\mathbf{f_k}\|_{0,\alpha;D}\right). \end{aligned}$$

Therefore, $(\mathbf{u}_\nu)$ converges to a vector field $\mathbf{u} \in [C^{1,\alpha}(\bar{D})]^m$ which is also a weak solution to the boundary value problem (A.11). By assumption this solution is uniquely defined. ■

The regularity result for a weak solution to an elliptic system of form (A.11) is now a simple consequence of Theorem A.10.

Corollary A.11 *Let $D \subset \mathbb{R}^2$ be a bounded domain of class $C^{1,\alpha}$, $\mathbf{g} \in [L^\infty(D)]^m$, $\mathbf{f_k} \in [C^\alpha(\bar{D})]^m$ ($k = 1, \ldots, m$), $\phi \in [C^{1,\alpha}(\bar{D})]^m$ and $\mathbf{u} \in [H^1(D)]^m$ a bounded weak solution to the boundary value problem (A.11). Then there holds $\mathbf{u} \in [C^{1,\alpha}(\bar{D})]^m$.*

Proof: We will rewrite equation (A.1) as $L\mathbf{u} = 0$. Choose $\sigma \in \mathbb{R}$ so large that the boundary value problem

$$\begin{aligned} (L - \sigma)\mathbf{u} &= 0 \qquad \text{in } D, \\ u &= 0 \qquad \text{on } \partial D \end{aligned}$$

only has the trivial solution in $[H^1(D)]^m$. Then, by Theorem A.10, the boundary value problem

$$\begin{aligned} (L - \sigma)\mathbf{v} &= \mathbf{g} + \operatorname{div} \mathbf{f_k} - \sigma\mathbf{u} \qquad \text{in } D, \\ v &= \phi \qquad \text{on } \partial D \end{aligned}$$

has a unique solution $\mathbf{v} \in [C^{1,\alpha}(\bar{D})]^m$. But $\mathbf{u}$ is also a solution to this system by assumption, so $\mathbf{u} = \mathbf{v}$ must hold. ∎

More technical arguments have to be employed to prove a similar result for boundary portions of class $C^{1,\alpha}$:

Theorem A.12 *Let T be a $C^{1,\alpha}$-boundary portion of a domain $D \in \mathcal{T}_{\alpha,\kappa_0,\delta,M}$, $\mathbf{g} \in [L^\infty(D)]^m$, $\mathbf{f_k} \in [C^\alpha(\bar{D})]^m$ ($k = 1, \ldots, m$), $\phi \in [C^{1,\alpha}(\bar{D})]^m$ and $\mathbf{u} \in [H^1(D)]^m$ a weak solution to the system (A.1) satisfiying $\mathbf{u} = \phi$ on T. Then $\mathbf{u} \in [\mathcal{V}^{1,\alpha}(D \cup T)]^m$ and for $D' \subset\subset D \cup T$ with $D' \in \mathcal{D}_{\alpha,\kappa_0,\delta,M}$, there holds*

$$\|\mathbf{u}\|_{1,\alpha;D'} \leq C \left(\|\mathbf{u}\|_{\infty;D} + \|\phi\|_{1,\alpha;D} + \|\mathbf{g}\|_{\infty;D} + \sum_{k=1}^{m} \|\mathbf{f_k}\|_{0,\alpha;D} \right),$$

where the constant C depends only on m, α, λ, Λ, κ_0, δ, M and $\operatorname{dist}(D', \partial D \setminus T)$.

Proof: The proof broadly follows that of Lemma 6.18 in [28]. Without loss of generality assume $\phi \equiv 0$. Choose $\mathbf{x_0} \in T$ arbitrarily. Then there exist a neighbourhood $T' \subset\subset T$ of $\mathbf{x_0}$ and a domain D^* so that $T' \subseteq \partial D$, $D^* \subset D$ of class $C^{1,\alpha}$ and so small, that Theorem A.10 can be applied.

The values of $\mathbf{u}$ on ∂D^* can now be extended to a vector field $\mathbf{v} \in [C(D') \cap C^{1,\alpha}(\bar{B})]^m$ where D' is chosen such that $D^* \subset\subset D'$ and $B := B_\rho(\mathbf{x_0}) \subset\subset D'$ for some $\rho > 0$ (see Lemma 6.38 and the subsequent remark in [28]). Let $(\mathbf{v}_\nu)$ be a sequence in $[C^{1,\alpha}(\overline{D^*})]^m$ that converges to $\mathbf{v}$ in $C(\overline{D^*})$ and also satisfies $\|\mathbf{v}_\nu\|_{1,\alpha;B} \leq C\|\mathbf{v}\|_{1,\alpha;B}$. Theorem A.10 garanties that the systems

$$\begin{aligned} L_k\mathbf{u} &= g_k + \operatorname{div} \mathbf{f_k} \qquad \text{in } D, \\ \mathbf{u} &= \mathbf{v}_\nu \qquad \text{on } \partial D \end{aligned}$$

have unique solution $\mathbf{u}_\nu \in [C^{1,\alpha}(\overline{D^*})]^m$. An application of Lemma A.6 now yields convergence of the sequence $(\mathbf{u}_\nu)$ in $[C^{1,\alpha}(\overline{B_{\rho/2}(\mathbf{x}_0) \cap D})]^m$. So $\mathbf{u} \in [\mathcal{V}^{1,\alpha}(D \cup T)]^m$ follows.

Now assume $D' \subset\subset D \cup T$, $D' \in \mathcal{D}_{\alpha,\kappa_0,\delta,M}$. As $T'' := \partial D' \cap T$ is bounded, setting $\rho := \operatorname{dist}(D', \partial D \setminus T)$, there exists a finite set of open balls $\{B_i := B_{\rho/2}(\mathbf{x}_i) : \mathbf{x}_i \in T'', i = 1, \ldots, N\}$ that covers T'' and, by applying Corollary A.9, in each ball we have the estimate

$$\|\mathbf{u}\|_{1,\alpha;B_i} \leq C \left(\|\mathbf{u}\|_{\infty;D} + \|\mathbf{g}\|_{\infty;D} + \sum_{k=1}^{m} \|\mathbf{f_k}\|_{0,\alpha;D} \right), \tag{A.13}$$

with the asserted dependance of the constant C on the parameters. Also, there exists some $\varepsilon > 0$ such that, setting $D'' := \{\mathbf{x} \in D' : \operatorname{dist}(\mathbf{x}, T'') > \varepsilon\}$, the union of D'' and the B_i covers all of D'. Then Lemma A.5 yields the estimate (A.13) in D'' and combining the two completes the proof. ■

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