On Multivariate Chebyshev Polynomials and Spectral Approximations on Triangles

Brett N. Ryland and Hans Z. Munthe-Kaas

Abstract In this paper we describe the use of multivariate Chebyshev polynomials in computing spectral derivations and Clenshaw–Curtis type quadratures. The multivariate Chebyshev polynomials give a spectrally accurate approximation of smooth multivariate functions. In particular we investigate polynomials derived from the A_2 root system. We provide analytic formulas for the gradient and integral of A_2 bivariate Chebyshev polynomials. This yields triangular based Clenshaw–Curtis quadrature and spectral derivation algorithms with $\mathscr{O}(N \log N)$ computational complexity. Through linear and nonlinear mappings, these methods can be applied to arbitrary triangles and non-linearly transformed triangles. A MATLAB toolbox and a C++ library have also been developed for these methods.

1 Introduction

Classical Chebyshev polynomials are among the most important building blocks in approximation theory. We recall some of their beautiful and immensely useful properties:

- Polynomial interpolation in Chebyshev zeros and Chebyshev extremal points converges exponentially fast for analytic functions on $[-1,1]$ and with superalgebraic speed for smooth functions.
- The Lebesgue constant for Chebyshev interpolation grows logarithmically in the number of interpolation points N.
- Chebyshev polynomials are orthogonal both with a continuous weighted inner product and also with discrete inner products based on Gauss-Chebyshev or Gauss-Chebyshev-Lobatto quadrature (nodes in Chebyshev zeros or Chebyshev extremal points).
- Chebyshev polynomial interpolation is equivalent to discrete Fourier cosine transform under a change of variables, thus all basic operations can be computed in $\mathscr{O}(N \log N)$ operations using FFT. This includes transforms between nodal values and expansion coefficients, mesh refinement and coarsening, products, integration and derivations.
- Integration and derivation of Chebyshev polynomial expansions can be done exactly. This leads to the highly accurate Clenshaw–Curtis quadrature and spectral Chebyshev derivation.

Chebyshev polynomials are also frequently used for multidimensional approximations. The standard approach is to construct multivariate polynomials as tensor products of univariate polynomials. However, this approach limits the application of multivariate Chebyshev approximations to rectangular and brick shaped domains, and domains that can be constructed from these by e.g., spectral elements.

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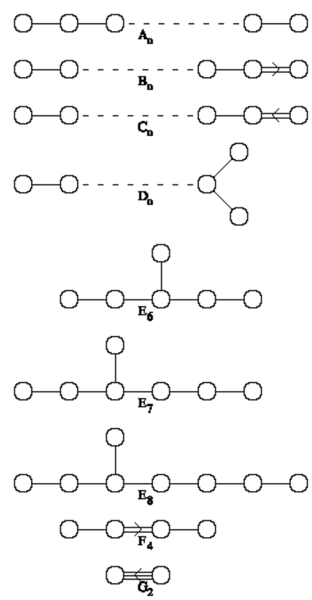
In this paper we will discuss families of multivariate Chebyshev polynomials obtained by an alternative construction, where tensor products of univariate polynomials appears as just one particular case. The construction is based on central ideas in group theory and representation theory. The construction was first done for particular cases by Koornwinder [12, 13] and later in full generality by Hoffman and Withers [10], see also [1, 5, 15]. Applications of these polynomials in numerical algorithms were discussed in [17]. For an introduction to the group theoretic background of this paper we refer to [6, 11].

This work is in particular motivated by the goal to construct spectral type discretisations on domains subdivided into triangles and simplices. Approximation theory on triangles have been discussed in various contexts, see [2, 3, 4, 7, 8, 9, 18, 20, 22]. Fast Fourier type transforms for the symmetric functions that appear in the context of multivariate Chebyshev polynomials are discussed in [16, 19].

The construction of multivariate Chebyshev polynomials starts by looking at periodic exponential functions through a kaleidoscope of mirrors acting on $\mathbb{R}^d$. More specifically we ask:

Which polytopes $S \subset \mathbb{R}^d$ generate a periodic tessellation of $\mathbb{R}^d$ under reflections about its faces?

Up to group isomorphisms there is just one such S for $d = 1$, four for $d = 2$ and seven for $d = 3$. For $d = 1$ we can take the domain $S = [0,\pi] \subset \mathbb{R}$, which generates a 2π periodic tessellation. For $d = 2$ the four possibilities are S being a rectangle or a triangle with 60^o–60^o–60^o, 45^o–45^o–90^o or 30^o–60^o–90^o angles. In $d = 3$, the possible polytopes are prisms with base polygon being one of the four 2-d cases, as well as three particular tetrahedra. For each of these polytopes, there exists a family of multivariate Chebyshev polynomials, which are orthogonal on a domain which is the image of S under a certain change of variables. The classification of all these polytopes S, and thus also the classification of multivariate Chebyshev polynomials, is done in terms of Dynkin diagrams. This is a graph with d nodes, each node representing one mirror in $\mathbb{R}^d$. The nodes are not connected if the mirrors are orthogonal, connected with one edge if the mirrors meet at 60^o, two edges if they meet at 45^o and three edges for 30^o. If two sets of nodes are totally disconnected, the mirrors form two subsets which are mutually orthogonal. In this case the corresponding polynomials become tensor products of each of the connected components. Thus, it is sufficient to classify only the *connected* Dynkin diagrams. The complete classification of connected Dynkin diagrams is shown on the right.



The classical univariate polynomials are represented by the diagram consisting of a single dot, A_1. Tensor product polynomials in d dimensions is represented as a diagram of d separate dots. The 2-d triangles 60^o–60^o–60^o, 45^o–45^o–90^o and 30^o–60^o–90^o are given by the diagrams A_2, B_2 and G_2.

Given such a domain S, we find a domain of periodicity P such that $S \subset P \subset \mathbb{R}^d$. The construction of multivariate Chebyshev polynomials goes as follows, exemplified by the classical case:

1. Take the Fourier basis functions on P.
   Classical: $S = [0,\pi]$, $P = [-\pi,\pi]$, Fourier basis $\exp(ik\theta)$.
2. Using reflection symmetries, fold the Fourier basis to symmetric functions on S.
   Classical: $\cos(k\theta) = \frac{1}{2}(\exp(ik\theta) + \exp(-ik\theta))$.
3. Use the symmetrised generators for the Fourier basis to change variables. This turns the symmetrised Fourier basis into Chebyshev polynomials on a transformed domain $\widetilde{S}$.
   Classical: $x = \frac{1}{2}(\exp(i\theta) + \exp(-i\theta)) = \cos(\theta)$, $\widetilde{S} = [-1,1]$.

It should be remarked that these polynomials enjoy most of the beautiful properties of their univariate cousins. However, in the classical case the transformed domain $\widetilde{S}$ is still an interval, whereas in the general case S is typically a simplex or a product of simplices, e.g., a prism, while the transformed domain $\widetilde{S}$ is a more complicated domain which is usually non-convex and often has cusps in the corners. Coping with the shape of $\widetilde{S}$ is the main difficulty in the practical use of the multivariate Chebyshev polynomials.

This paper is organised as follows. In Section 2 we review basic properties of root systems and multivariate Chebyshev polynomials with special emphasis on the A_2 case related to symmetries of the equilateral triangle. Section 3 treats new spectrally accurate methods for computing gradients on triangles. Section 4 discusses Clenshaw–Curtis type quadratures for triangles. Lastly, in Chapter 5 we demonstrate the algorithms through numerical experiments.

2 Chebyshev polynomials and root systems

In this Section we will review the basic definitions and properties of multivariate Chebyshev polynomials. Root systems give explicit information about the reflections and translations discussed in the introduction, and are hence important for computational algorithms.

2.1 Root Systems

A root system Φ on a vector space V is a collection of vectors satisfying the following four conditions [11]:

(i) Φ spans V,
(ii) If $\alpha \in \Phi$, then $c\alpha \in \Phi \iff c \in \{1, -1\}$,
(iii)If $\alpha \in \Phi$, then Φ is closed under the reflection $\sigma_\alpha = I - 2\frac{\alpha\alpha^T}{\alpha^T\alpha}$,
(iv)Integrality condition: $\alpha, \beta \in \Phi \implies <\beta, \alpha>= 2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$, where $(\alpha, \beta) = \alpha^T\beta$.

In 1 dimension, the only root system is given by $\Phi = \{\alpha, -\alpha\}$, where α is a non-zero vector. In 2 dimensions, there are four root systems, which are shown in Figure 1. The first of these is a tensor product of the

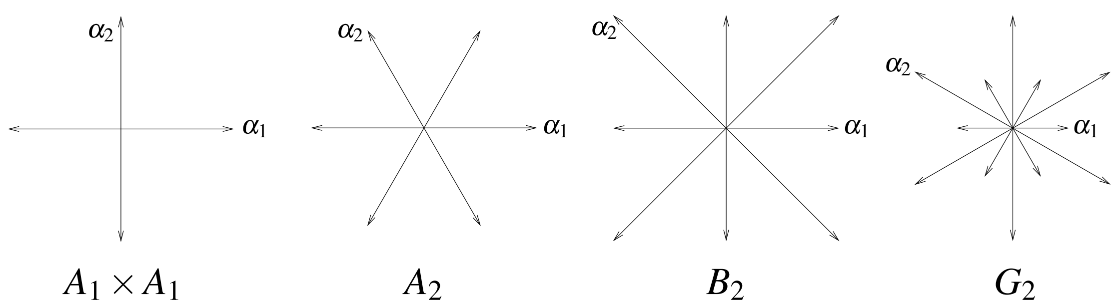


Fig. 1 Root systems in two dimensions.

1-dimensional root system, while the remainder are irreducible. A complete classification of irreducible root systems is given by the Dynkin diagrams A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4 and G_2.

2.2 Multivariate Chebyshev Polynomials

Following [17], let Φ be a root system with a basis $\{\alpha_1, \ldots, \alpha_d\}$ where the α_i are simple positive roots of Φ. Corresponding to this root system is the Weyl group $W = \{\sigma_\alpha | \alpha \in \Phi\}$, where σ_α is the reflection in the hyperplane orthogonal to α. Expressed in the basis $\{\alpha_j\}$, W is generated by the integer matrices

$$\widetilde{\sigma}_i = I - e_i e_i^T C, \qquad i = 1, \ldots d,$$

where C is the Cartan matrix $C_{jk} = 2\frac{\alpha_j^T \alpha_k}{\alpha_j^T \alpha_j}$, I is the identity matrix, and $\{e_i\}$ is the standard basis on $\mathbb{R}^d$.

The *root lattice* is the set of all translations generated by the roots $\Lambda = \mathrm{span}_{\mathbb{Z}}\{\alpha_1, \dots, \alpha_d\}$, and the *affine Weyl group* $\widetilde{W} = \Lambda \rtimes W$ is the group generated by both translations and reflections along the roots. The fundamental domain of $\widetilde{W}$ is a polytope $S \subset \mathbb{R}^d$ and the periodicity domain P is the parallelepiped spanned by the simple positive roots α_j.

In terms of the basis $\frac{1}{2\pi}\{\alpha_1/|\alpha_1|, \dots, \alpha_d/|\alpha_d|\}$, we identify P with the abelian group $G = (\mathbb{R}/2\pi\mathbb{Z})^d$, whose dual group is $\widehat{G} = \mathbb{Z}^d$. The Fourier basis for periodic functions is defined via the pairing

$$(k, \theta) := \mathrm{e}^{\mathrm{i}k\cdot\theta},$$

where $k \in \widehat{G}$ and $\theta \in G$. Via the symmetries of the Weyl group, one can define the symmetrised pairing

$$(k, \theta)_s := \frac{1}{|W|}\sum_{g\in W}(k, g\theta) = \frac{1}{|W|}\sum_{g\in W}(g^T k, \theta),$$

where $|W|$ is the number of elements in the Weyl group W.

Note that in the same way that G can be recovered (up to periodicity) from the fundamental domain by application of elements from the Weyl group, one can define a dual fundamental domain (i.e., a fundamental domain in $\widehat{G}$) such that $\widehat{G}$ is recovered from the dual fundamental domain by application of the transpose of elements from the Weyl group.

We can now define the multivariate Chebyshev polynomials in the following way:

Definition 1 *The multivariate Chebyshev polynomial of degree k is given by*

$$T_k(z) := (k, \theta)_s, \tag{1}$$

where $z_j(\theta) = (e_j, \theta)_s$ for $j = 1, \dots, d$, and the e_j are the standard basis vectors in $\mathbb{R}^d$.

The multivariate Chebyshev polynomials are related to each other via the following relations

$$\begin{gathered} T_0 = 1, \\ T_{e_j} = z_j, \\ T_k = T_{g^T k} \text{ for } g \in W, \\ T_{-k} = \overline{T_k}, \\ T_k T_l = \frac{1}{|W|}\sum_{g\in W} T_{k+g^T l} = \frac{1}{|W|}\sum_{g\in W} T_{l+g^T k}. \end{gathered} \tag{2}$$

These relations clearly show that the multivariate Chebyshev polynomials are indeed polynomials in the z_j.

A multivariate function may be expanded in an infinite weighted sum over multivariate Chebyshev polynomials,

$$f(z) = \sum_{k\in\widehat{G}} a(k) T_k(z),$$

where the coefficients $a(k)$ can be obtained via a Fourier transform, i.e.,

$$\begin{aligned} \sum_{k\in\widehat{G}} a(k) T_k(z) &= \frac{1}{|W|}\sum_{k\in\widehat{G}} a(k) \sum_{g\in W}(g^T k, \theta) \\ &= \frac{1}{|W|}\sum_{k\in\widehat{G}}\sum_{g\in W} a(g^T k)(g^T k, \theta) \\ &= \sum_{k\in\widehat{G}} a(k)(k, \theta) \end{aligned}$$

since $a(k) = a(g^T k)$ for all $g \in W$. Thus $a(k) = \widehat{f_s}(\theta)$, where $f_s(\theta)$ is the pullback of $f(z)$ to the periodicity domain P. Note that $f_s(\theta)$ is a symmetric function, $f_s(g\theta) = f_s(\theta)$ for all $g \in W$.

To do numerical computations, we discretise P with a regular lattice and sample $f(z)$. It is essential that this lattice respects all the symmetries of $\widetilde{W}$. There are several ways to accomplish this. One possibility is to downscale the root lattice Λ by a factor N so that P contains an $N\times N$ grid. This grid is invariant under the action of $\widetilde{W}$. Thus we obtain a finite polynomial approximation

$$f(z) \approx P_N(z) = \sum_{k\in\widehat{G}_N} a_N(k)T_k(z), \tag{3}$$

where $\widehat{G}_N = (\mathbb{Z}/N\mathbb{Z})^d$ is the d-dimensional N-periodic integer lattice. If the approximating polynomial, $P_N(z)$, is evaluated at the set of points $\{z^* = z(\theta_j)\}$, where $j\in\widehat{G}_N$ and $\theta_j = 2\pi j/N \in \mathbb{R}^d$, then the expansion of $P_N(z)$ is

$$P_N(z(\theta_j)) = \sum_{k_d=0}^{N-1}\cdots\sum_{k_1=0}^{N-1} a_N(k)\exp(\mathrm{i}k\cdot\theta_j),$$

which is simply an unnormalised d-dimensional inverse discrete Fourier transform. Thus, the $a_N(k)$ are given by

$$a_N(k) = \frac{1}{N^d}\sum_{j_d=0}^{N-1}\cdots\sum_{j_1=0}^{N-1} P_N(z(\theta_j))\exp(\mathrm{i}k\cdot\theta_j), \tag{4}$$

which can be computed quickly via the d-dimensional fast Fourier transform $\mathscr{F}_d$,

$$\{a_N(k)\} = \frac{1}{N^d}\mathscr{F}_d\{P_N(z^*)\}.$$

In the sequel, we omit the subscript N and write just $a(k)\equiv a_N(k)$.

2.3 The A_2 root system

The Weyl group for the A_2 root system is

$$W = \left\{ \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}-1&1\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\1&-1\end{bmatrix}, \begin{bmatrix}-1&1\\-1&0\end{bmatrix}, \begin{bmatrix}0&-1\\1&-1\end{bmatrix}, \begin{bmatrix}0&-1\\-1&0\end{bmatrix} \right\}. \tag{5}$$

Here the fundamental domain is that of the $\widetilde{A}_2$ affine Weyl group, which is an equilateral triangle in the plane (see Figure 2), and the dual fundamental domain is the set of points in Fourier space given by $\{k\in(\mathbb{Z}/N\mathbb{Z})^2 | k_2\le k_1, 2k_1+k_2\le N\}$.

With this Weyl group, Eq. (2) gives rise to the following recursion formulas

$$\begin{gathered} T_{0,0}=1, \quad T_{1,0}=z_1, \quad T_{0,1}=z_2=T_{-1,0}=\overline{z_1}, \\ T_{n,0}=3z_1T_{n-1,0}-3z_2T_{n-2,0}+T_{n-3,0}, \\ T_{n,m}=\frac{1}{2}(3T_{n,0}\overline{T_{m,0}}-T_{n-m,0}). \end{gathered}$$

However, it is more convenient, in practice, to work in the real-valued coordinates

$$\begin{aligned} x_1 &= \frac{1}{2}(z_1+z_2) = \frac{1}{3}(\cos(\theta_1)+\cos(\theta_2)+\cos(\theta_1-\theta_2)), \\ x_2 &= \frac{1}{2\mathrm{i}}(z_1-z_2) = \frac{1}{3}(\sin(\theta_1)-\sin(\theta_2)-\sin(\theta_1-\theta_2)). \end{aligned}$$

Clearly, the multivariate Chebyshev polynomials are also polynomials in these coordinates.

While the multivariate Chebyshev polynomials are defined in a unit cell of the lattice, they exist in θ coordinates as multiple images (not necessarily whole) of the fundamental domain. For the A_2 root system,

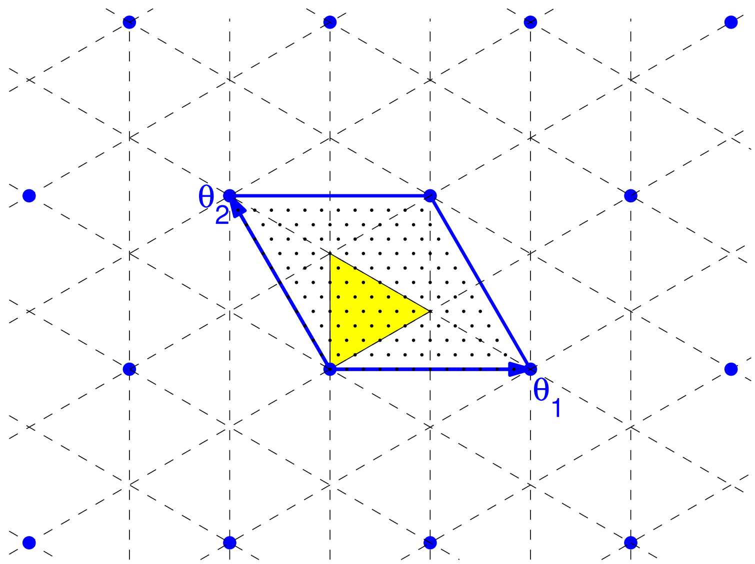


Fig. 2 The A_2 root system showing fundamental domain (yellow triangle), translation group (blue hexagonal lattice) and downscaled lattice (small black dots) for the downscaling factor $N = 12$. The blue arrows indicate the θ coordinates.

this is the equilateral triangle on the right of Figure 3. However, in the x coordinates, this triangle is mapped to a deltoid as shown on the left of Figure 3.

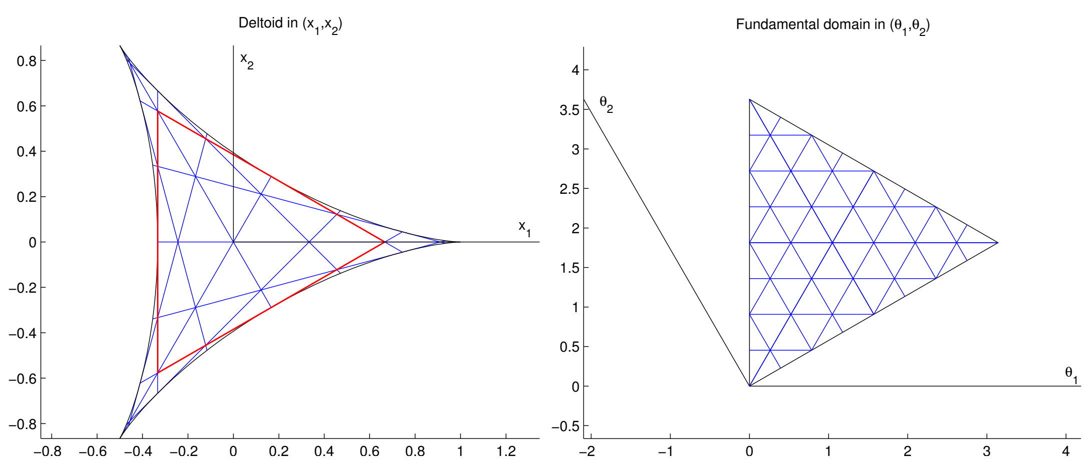


Fig. 3 $N = 12$ equally spaced tangent lines to the deltoid in x and θ coordinates. The points $\{x^*(\theta)\}$ lie at the intersection of these lines, cf. Figure 2.

Lemma 2.1 *The Lebesgue constant $\lambda(N)$ for the points x^* grows as $\mathscr{O}((\log(N))^2)$.*

In higher dimensions the Lebesgue constant grows as $\mathscr{O}((\log(N))^d)$. The proof of this result relies on properties of the multidimensional Dirichlet kernel, see also [14].

3 Computing gradients

Let us consider the gradient of the approximating function $P_N(z)$,

$$\nabla_z f(z) \approx \nabla_z P_N(z) = \sum_{k\in\widehat{G}_N} a(k)\nabla_z T_k(z). \tag{6}$$

For the univariate Chebyshev polynomials, the derivative of the approximating polynomial can be written exactly as a weighted sum over Chebyshev polynomials via the relation,

$$\partial_z P_N(z) = \sum_{k=0}^{N} a(k)\partial_z T_k(z) = \sum_{k=0}^{N-1} b(k)T_k(z), \tag{7}$$

where the $b(k)$ are calculated recursively by

$$b(k-1) = b(k+1) + 2ka(k) \qquad \text{for } k = N-1,\dots,1, \tag{8}$$

with $b(N+1) = b(N) = 0$. This recursion formula can be obtained by substituting the relation

$$2T_k(z) = \frac{\partial_z T_{k+1}(z)}{k+1} - \frac{\partial_z T_{k-1}(z)}{k-1},$$

into Eq. (7) and matching terms.

In the general multivariate case the coefficients of the gradient are vectors $b(k) \in \mathbb{C}^d$. It can be shown that these satisfy the recursion

$$|W|ka(k) = \sum_{l=1}^{d}\sum_{\gamma\in W} b(k-\gamma^T e_l)_l(\gamma^T e_l). \tag{9}$$

The $b(m)$ in Eq. (9) can be obtained by setting $b(M) = 0$ and $a(M) = 0$ for M outside the dual fundamental domain and iteratively determining the $b(m)$ as m approaches the origin.

3.1 The A_2 root system

As mentioned earlier, we prefer to work in the real-valued (x_1,x_2) coordinates when dealing with the A_2 root system. The effect of this is that the gradient of the approximating polynomial becomes

$$\nabla_x P_N(x) = J_x(z)^{-T} J_z(\theta)^{-T} \sum_{k\in\widehat{G}_N} a(k)\nabla_\theta T_k(\theta) = J_x(z)^{-T} \sum_{m\in\widehat{G}_N} b(m)T_m(\theta), \tag{10}$$

where $J_z(\theta)$ is the Jacobian of the transformation between the θ and z domains, and

$$J_x(z)^{-T} = \begin{bmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{bmatrix}.$$

The recursion formula (Eq. (9)) can then be used to determine the $b(m)$ from the $a(k)$ via the following 3 steps

1. Sort the k in the dual fundamental domain by $k_1 + k_2 = c$ for $c = 0,1,\dots$, and with increasing k_2 for each c (see Figure 4).
2. Assume that $b(k) = 0$ for k not in the dual fundamental domain and apply the stencils in Figure 5 to each of the k in the fundamental domain in the reverse order to the sorting in step 1.

   a) If $k_2 > 0$, use stencil a) to obtain $b(k_1,k_2)_1$:

$$b(k_1,k_2)_1 = 3(k_1+1)a(k_1+1,k_2) + b(k_1+2,k_2-1)_1 + b(k_1+2,k_2)_2 - b(k_1,k_2+1)_2$$

otherwise, use stencil b):

$$b(k_1,0)_1 = 3(k_1+1)a(k_1+1,0) + b(k_1+1,1)_1 + b(k_1+2,0)_2 - b(k_1,1)_2.$$

b) If $k_1 > 0$, use stencil c) to obtain $b(k_1,k_2)_2$:

$$b(k_1,k_2)_2 = 3(k_2+1)a(k_1,k_2+1) + b(k_1-1,k_2+2)_2 + b(k_1,k_2+2)_1 - b(k_1+1,k_2)_1$$

otherwise use stencil d):

$$b(0,0)_2 = 3a(0,1) + b(1,1)_2 + b(0,2)_1 - b(1,0)_1.$$

c) Apply the conjugate symmetry:

$$b(k_2,k_1)_1 = \overline{b(k_1,k_2)_2},$$
$$b(k_2,k_1)_2 = \overline{b(k_1,k_2)_1}.$$

3. Apply the symmetries from the symmetry group W to obtain the rest of the coefficients:

$$b(g^T k) = b(k) \quad \text{for } g \in W.$$

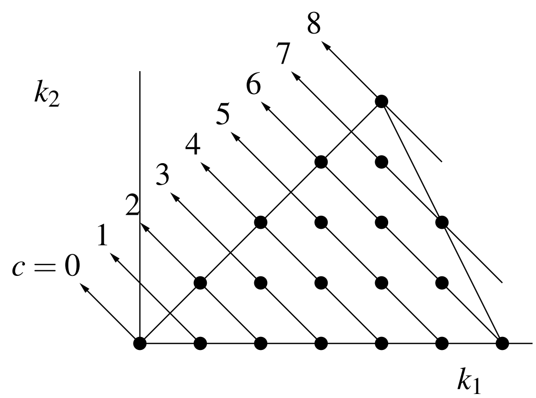


Fig. 4 Sorted dual fundamental domain for the A_2 root system with $N = 12$.

By using this recursion formula, one can calculate the gradient of a function in $\mathscr{O}(N^2 + \alpha(N^2 \log N^2))$ time, where α indicates the amount of the time spent performing the fast Fourier transform and its inverse. Note that there are $\mathscr{O}(N^2)$ sample points in the computational domain, thus in terms of the number of sample points, the computational complexity of this approach is only marginally greater than linear, the main cost being the FFT.

4 Clenshaw–Curtis quadrature

Clenshaw–Curtis quadrature is a well-known technique for univariate integration. For a function on $[-1,1]$ the method amounts to approximating a given function by a finite Chebyshev expansion and integrating this polynomial exactly. For smooth functions this method behaves nearly as good as Gaussian quadrature [21].

As with the univariate Chebyshev polynomials, the integral of a multivariate function may be evaluated rapidly with a multivariate Clenshaw–Curtis quadrature technique.

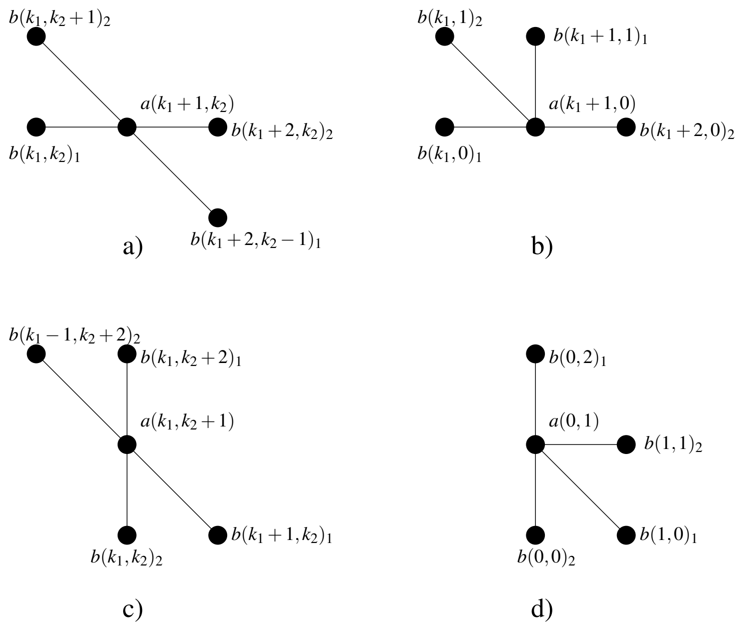


Fig. 5 Stencils for use in step 2.

$$\int_{\Omega_z} f(z)\mathrm{d}z \approx \int_{\Omega_z} P_N(z)\mathrm{d}z = \sum_{k\in\widehat{G}_N} a(k) \int_{\Omega_\theta} T_k(\theta)|J_z(\theta)|\mathrm{d}\theta, \tag{11}$$

where $|J_z(\theta)|$ is the absolute value of the determinant of the Jacobian and Ω_z and Ω_θ are the fundamental domains in z and θ coordinates respectively.

Since both $T_k(\theta)$ and the Jacobian $J_z(\theta)$ have terms of the form (k,θ) as building blocks, we can expand the integrals in Eq. (11) as

$$\int_{\Omega_\theta} T_k(\theta)|J_z(\theta)|\mathrm{d}\theta = \sum_{\kappa\in\widehat{G}_N} b(\kappa) \int_{\Omega_\theta} (\kappa,\theta)\mathrm{d}\theta, \tag{12}$$

for some $b(\kappa)$ that is only non-zero near $\kappa = k$. Furthermore, these integrals need only be computed once.

4.1 The A_2 root system

For the A_2 root system, the determinant of the Jacobian $J_x(\theta)$ is

$$\begin{aligned}\Gamma &= \frac{1}{9}(\sin(\theta_1+\theta_2)+\sin(\theta_1-2\theta_2)-\sin(2\theta_1-\theta_2)) \\ &= -\frac{\mathrm{i}}{2}\cdot\frac{1}{9}\Bigg(\left(\begin{bmatrix}1\\1\end{bmatrix},\theta\right)-\left(\begin{bmatrix}-1\\-1\end{bmatrix},\theta\right)+\left(\begin{bmatrix}1\\-2\end{bmatrix},\theta\right) \\ &\quad -\left(\begin{bmatrix}-1\\2\end{bmatrix},\theta\right)+\left(\begin{bmatrix}-2\\1\end{bmatrix},\theta\right)-\left(\begin{bmatrix}2\\-1\end{bmatrix},\theta\right)\Bigg),\end{aligned}$$

which is zero on the boundary of the fundamental domain and negative within it. Furthermore, the orientation of the deltoid in the x coordinates is opposite to that of the fundamental domain in θ coordinates. Thus, Eq. (12) becomes

$$\int_{\Omega_\theta} T_k(\theta)|J_x(\theta)|\mathrm{d}\theta = \frac{\mathrm{i}}{2}\cdot\frac{1}{9}\cdot\frac{1}{|W|}\sum_{\kappa\in S}\delta_\kappa \int_{\Omega_\theta}(\kappa,\theta)\mathrm{d}\theta \tag{13}$$

for the set S consisting of $g^T k + l$, where $g \in W$,

$$l \in \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\},$$

and

$$\delta_\kappa = \begin{cases} 1 & \text{if } l \in \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}, \\ -1 & \text{if } l \in \left\{ \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}. \end{cases}$$

The fundamental domain in θ coordinates is the region bounded by

$$\begin{aligned} \theta_1 + \theta_2 &= 2\pi, \\ \theta_1 &= 2\theta_2, \\ \theta_2 &= 2\theta_1. \end{aligned}$$

Therefore, the integral $\int_{\Omega_\theta}(\kappa,\theta)\mathrm{d}\theta$ becomes

$$\int_{\Omega_\theta}(\kappa,\theta)\mathrm{d}\theta = \int_0^{\frac{2\pi}{3}}\int_{\frac{1}{2}\theta_2}^{2\theta_2}(\kappa,\theta)\mathrm{d}\theta_1\mathrm{d}\theta_2 + \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}\int_{\frac{1}{2}\theta_2}^{2\pi-\theta_2}(\kappa,\theta)\mathrm{d}\theta_1\mathrm{d}\theta_2,$$

which evaluates to

$$\int_{\Omega_\theta}(\kappa,\theta)\mathrm{d}\theta = \begin{cases} \frac{2\pi^2}{3} & \text{if } \kappa_1 = \kappa_2 = 0, \\ \frac{4\pi\mathrm{i}}{3\kappa_1} & \text{if } \kappa_1 \neq 0, \kappa_2 + \frac{1}{2}\kappa_1 = 0, \\ -\frac{2\pi\mathrm{i}}{3\kappa_1} & \text{if } \kappa_1 \neq 0, \kappa_2 + 2\kappa_1 = 0, \\ -\frac{2\pi\mathrm{i}}{3\kappa_1} & \text{if } \kappa_1 \neq 0, \kappa_2 - \kappa_1 = 0, \\ \frac{3}{2\kappa_2^2}(-(\kappa_2,\frac{4\pi}{3}) + 2(\kappa_2,\frac{2\pi}{3}) - 1) & \text{if } \kappa_1 = 0, \kappa_2 \neq 0, \\ -\frac{1}{\kappa_1}(\frac{1}{\kappa_2+2\kappa_1}((\kappa_2+2\kappa_1,\frac{2\pi}{3})-1) & \\ \quad -\frac{1}{\kappa_2+\frac{1}{2}\kappa_1}((\kappa_2+\frac{1}{2}\kappa_1,\frac{4\pi}{3})-1) & \\ \quad\quad +\frac{1}{\kappa_2-\kappa_1}(\kappa_2-\kappa_1,\frac{2\pi}{3})((\kappa_2-\kappa_1,\frac{2\pi}{3})-1)) & \text{otherwise.} \end{cases} \tag{14}$$

Note that due to the symmetries in $\widehat{G}_N$, one need only evaluate $\int_{\Omega_\theta} T_k(\theta)|J_x(\theta)|\mathrm{d}\theta$ for values of k in the dual fundamental domain. The full integral (Eq. (11)) can then be computed by summing over just the k lying in the dual fundamental domain and weighting each $a(k)$ by the size of the orbit of k under the group action of W. Curiously, due to the symmetries of $T_k(\theta)$ and $|J_x(\theta)|$, we have the following lemma.

Lemma 4.1 *The integral* $\int_{\Omega_\theta} T_k(\theta)|J_x(\theta)|\mathrm{d}\theta$ *is zero unless* $k_1 = k_2$ *or* $|k_1 - k_2| = 3$.

Proof. First, let us rewrite Eq. (13) as

$$\begin{aligned} \int_{\Omega_\theta} T_k(\theta)|J_x(\theta)|\mathrm{d}\theta &= \frac{\mathrm{i}}{18|W|}\sum_{\kappa\in S}\delta_\kappa\int_{\Omega_\theta}(\kappa,\theta)\mathrm{d}\theta \\ &= \frac{\mathrm{i}}{18|W|}\sum_m \delta_m \sum_{g\in W}\delta_g\int_{\Omega_\theta}(g^T m,\theta)\mathrm{d}\theta, \end{aligned}$$

where

$$m \in k + \eta^T l = \left\{ \begin{bmatrix} k_1+1 \\ k_2+1 \end{bmatrix}, \begin{bmatrix} k_1-1 \\ k_2+2 \end{bmatrix}, \begin{bmatrix} k_1-2 \\ k_2+1 \end{bmatrix}, \begin{bmatrix} k_1-1 \\ k_2-1 \end{bmatrix}, \begin{bmatrix} k_1+1 \\ k_2-2 \end{bmatrix}, \begin{bmatrix} k_1+2 \\ k_2-1 \end{bmatrix} \right\}$$

and $\delta_m \in \{1,-1,1,-1,1,-1\}$ respectively. The set $S = g^T m$ is then given by

$$S = \left\{ \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}, \begin{bmatrix} -m_1 \\ m_1+m_2 \end{bmatrix}, \begin{bmatrix} -m_1-m_2 \\ m_1 \end{bmatrix}, \begin{bmatrix} -m_2 \\ -m_1 \end{bmatrix}, \begin{bmatrix} m_2 \\ -m_1-m_2 \end{bmatrix}, \begin{bmatrix} m_1+m_2 \\ -m_2 \end{bmatrix} \right\} \tag{15}$$

with $\delta_g \in \{1,-1,1,-1,1,-1\}$ respectively, such that $\delta_\kappa = \delta_m \delta_g$.

Note that if $k_1 - k_2 \in \{-3,0,3\}$, then for some $\kappa \in S$ we have that $\kappa_1 = \kappa_2$, $\kappa_1 + 2\kappa_2 = 0$ or $2\kappa_1 + \kappa_2 = 0$. In this case, the first four lines of Eq. (14) play a part in the sum and the integral can be non-zero. On the other hand, if $k_1 - k_2 \notin \{-3,0,3\}$, then none of the first four lines of Eq. (14) play a part in the sum.

Let us consider the reduced sum

$$\sum_{g\in W} \delta_g \int_{\Omega_\theta} \left(g^T m, \theta\right) \mathrm{d}\theta. \tag{16}$$

Now, if one of the $\kappa \in g^T m$ in this sum is of the form $\kappa_1 = 0$, $\kappa_2 \neq 0$, then it can be easily seen from Eq. (15) that S reduces to three distinct pairs, each of which have δ_g being of opposite sign. Thus, the terms in Eq. (16) coming from the fifth line of Eq. (14) identically cancel.

It just remains to consider Eq. (16) where all of the integrals $\int_{\Omega_\theta} (g^T m, \theta) \mathrm{d}\theta$ are evaluated by the last line of Eq. (14). Directly evaluating this sum, one finds that after some simple but tedious manipulation, Eq. (16) reduces to

$$\sum_{g\in W} \delta_g \int_{\Omega_\theta} \left(g^T m, \theta\right) \mathrm{d}\theta = \frac{4\left(m_1^2 + m_1 m_2 + m_2^2\right)^2 \left(\overline{\alpha} - \alpha^2\right)}{m_1 m_2 \left(m_2^2 - m_1^2\right)\left(m_1 + 2m_2\right)\left(m_2 + 2m_1\right)},$$

where $\alpha = \left(\kappa_2 - \kappa_1, \frac{2\pi}{3}\right) = \left(\kappa_2 + 2\kappa_1, \frac{2\pi}{3}\right) = \overline{\left(\kappa_2 + \frac{1}{2}\kappa_1, \frac{4\pi}{3}\right)}$ and $\overline{\alpha} - \alpha^2 = 0$ since $\kappa_2 - \kappa_1 \in \mathbb{Z}$ for all k.

Thus, the integral $\int_{\Omega_\theta} T_k(\theta) |J_x(\theta)| \mathrm{d}\theta$ over the deltoid is zero unless $k_1 = k_2$ or $|k_1 - k_2| = 3$. $\square$

5 Triangles

The reader is no doubt aware that deltoids are not, in general, good shapes for decomposing surfaces into. Rather, it is much more desirable to decompose a surface into a number of triangles, which can then be mapped to a standard triangle either with linear or non-linear maps.

The difficulty now arises as to how to apply the multivariate Chebyshev polynomials (which naturally live on the deltoid) to such a triangle. We envisage two possible methods of doing this:

(I) Stretch the deltoid to the triangle with corners at the corners of the deltoid.
(II) Use the equilateral triangle that is inscribed in the deltoid.

Method (I) is appealing, since all of the data points in the deltoid lie within the triangle. Straightening maps are discussed in [17]. However, due to the shape of the deltoid, with its singularities at the corners, we are not able to straighten the deltoid to a triangle without compromising spectral convergence.

On the other hand, method (II) has the advantage that it does not require any further mappings (the gradient algorithm can be used directly and the Clenshaw–Curtis quadrature algorithm can be used with only a small modification to the integral of (k,θ) over the fundamental domain). However, this method makes use of the data points that lie outside the triangle to obtain the coefficients $a(k)$. This has implications for the use of these algorithms in spectral and spectral element methods on domains with boundaries, which will be discussed in a later paper.

5.1 Clenshaw–Curtis quadrature over a triangle

Since we are restricting the domain of integration, Ω_x, to the equilateral triangle inscribed within the deltoid (red triangle in Figure 3), the integrals $\int_{\Omega_\theta} (\kappa, \theta) \mathrm{d}\theta$ in Section 4 must be modified. They become

$$\int_{\Omega_\theta}(\kappa,\theta)\mathrm{d}\theta = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\int_{\frac{\pi}{3}}^{\theta_2+\frac{\pi}{3}}(\kappa,\theta)\mathrm{d}\theta_1\mathrm{d}\theta_2 + \int_{\frac{2\pi}{3}}^{\pi}\int_{\theta_2-\frac{\pi}{3}}^{\pi}(\kappa,\theta)\mathrm{d}\theta_1\mathrm{d}\theta_2,$$

where Ω_θ is the restriction of the fundamental domain to the equilateral triangle in θ coordinates.

As with the integral over the deltoid, this integral can be evaluated directly to give

$$\int_{\Omega_\theta}(\kappa,\theta)\mathrm{d}\theta = \begin{cases} \frac{\pi^2}{3} & \text{if } \kappa_1=\kappa_2=0, \\ \frac{\mathrm{i}\pi}{3\kappa_2}((\kappa_2,\frac{\pi}{3})-(\kappa_2,\pi))+\frac{1}{\kappa_2^2}(2(\kappa_2,\frac{2\pi}{3})-(\kappa_2,\pi)-(\kappa_2,\frac{\pi}{3})) & \text{if } \kappa_1=0, \kappa_2\neq 0, \\ \frac{\mathrm{i}\pi}{3\kappa_1}((\kappa_1,\frac{\pi}{3})-(\kappa_1,\pi))+\frac{1}{\kappa_1^2}(2(\kappa_1,\frac{2\pi}{3})-(\kappa_1,\pi)-(\kappa_1,\frac{\pi}{3})) & \text{if } \kappa_2=0, \kappa_1\neq 0, \\ \frac{2}{\kappa_1^2}(1-\cos(\kappa_1\frac{\pi}{3}))+\frac{2\pi}{3\kappa_1}\sin(\kappa_1\frac{\pi}{3}) & \text{if } \kappa_1+\kappa_2=0, \\ \frac{1}{\kappa_1\kappa_2}\big((\kappa_1,\frac{\pi}{3})((\kappa_2,\frac{2\pi}{3})-(\kappa_2,\frac{\pi}{3}))-(\kappa_1,\pi)((\kappa_2,\pi)-(\kappa_2,\frac{2\pi}{3}))\big) & \\ \quad+\frac{1}{\kappa_1(\kappa_2+\kappa_1)}\big((\kappa_1,\frac{2\pi}{3})((\kappa_2,\pi)+(\kappa_2,\frac{\pi}{3})) & \\ \qquad-(\kappa_2,\frac{2\pi}{3})((\kappa_1,\pi)+(\kappa_1,\frac{\pi}{3}))\big) & \text{otherwise.} \end{cases} \tag{17}$$

Again, the symmetries of $T_k(\theta)$ and $|J_x(\theta)|$ restrict the values of k for which the integral $\int_{\Omega_\theta}T_k(\theta)|J_x(\theta)|\mathrm{d}\theta$ is non-zero. We obtain the following lemma.

Lemma 5.1 *The integral* $\int_{\Omega_\theta}T_k(\theta)|J_x(\theta)|\mathrm{d}\theta$ *is zero unless* $k_1-k_2=0 \pmod 3$.

Proof. As with the proof of Lemma 4.1, we consider the reduced sum

$$\sum_{g\in W}\delta_g\int_{\Omega_\theta}(g^T m,\theta)\mathrm{d}\theta, \tag{18}$$

where m and δ_g are given in Lemma 4.1 and the integrals are evaluated by Eq. (17).

If $m_1=0$, $m_2=0$ or $m_1+m_2=0$, then it can be easily seen from Eq. (15) that the $g^T m$ occur in pairs with δ_g being of opposite sign. Thus the contributions to Eq. (18) from the first four lines of Eq. (17) identically cancel with each other and we need only consider the contributions coming from the last line of Eq. (17).

Directly evaluating Eq. (18) where the integrals are evaluated by the last line of Eq. (17) gives

$$\sum_{g\in W}\delta_g\int_{\Omega_\theta}(g^T m,\theta)\mathrm{d}\theta = \frac{2\mathrm{i}(m_1\alpha+m_2\beta)}{m_1m_2(m_1+m_2)},$$

where

$$\begin{aligned}\alpha = &-\sin(m_1\pi/3)+\sin((m_1-2m_2)\pi/3)+\sin((m_2+3m_1)\pi/3) \\ &+\sin((m_1+3m_2)\pi/3)-\sin((m_1+m_2)\pi/3)+\sin((3m_1+2m_2)\pi/3)\end{aligned}$$

and

$$\begin{aligned}\beta = &-\sin(m_2\pi/3)+\sin((m_2-2m_1)\pi/3)+\sin((m_1+3m_2)\pi/3) \\ &+\sin((m_2+3m_1)\pi/3)-\sin((m_2+m_1)\pi/3)+\sin((3m_2+2m_1)\pi/3).\end{aligned}$$

Now, let $a=m_2-m_1$, such that α and β become

$$\alpha = -\sin(m_1\pi/3)\,(1+2\cos(2a\pi/3))-\sin(2m_1\pi/3)\,(\cos(a\pi)+2\cos(a\pi/3))$$

and

$$\beta = -\sin(m_2\pi/3)\,(1+2\cos(2a\pi/3))-\sin(2m_2\pi/3)\,(\cos(a\pi)+2\cos(a\pi/3)),$$

which are only non-zero if $a=0 \pmod 3$.

Since $m_1-m_2=k_1-k_2 \pmod 3$, we find that the integral $\int_{\Omega_\theta}T_k(\theta)|J_x(\theta)|\mathrm{d}\theta$ over the triangle is only non-zero if $k_1-k_2=0 \pmod 3$. □

5.2 Nonlinear transformations

Given a nonlinear mapping $\phi : \mathbb{R}^2 \mapsto \mathbb{R}^2$ such that $y = \phi(x)$, the Jacobian of this map can be calculated numerically at each point $y(x)$ as

$$J_y(x) = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{bmatrix}$$

using the gradient algorithm of Section 3.1.

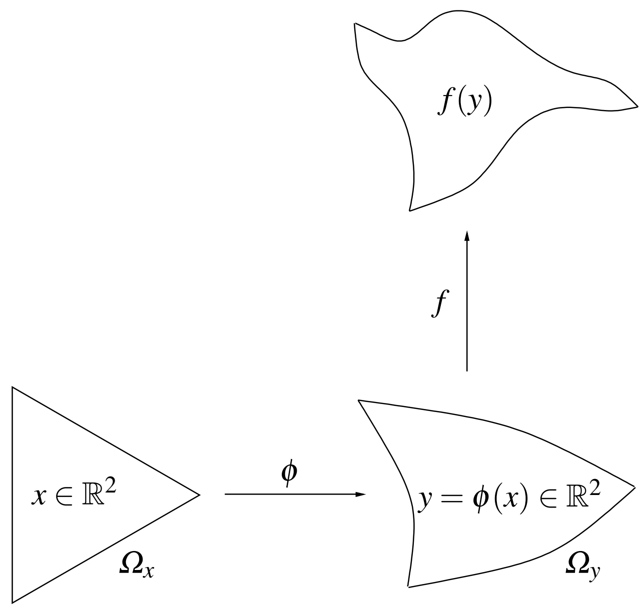


Fig. 6 A function $f(y)$ is defined on a nonlinear triangle Ω_y, which is mapped from the equilateral triangle Ω_x by the map ϕ.

The gradient of a function $f(y)$ on the nonlinear triangle can then be obtained by multiplying the gradient obtained from the gradient algorithm by the inverse transpose of the Jacobian at each point in the domain. I.e.,

$$\nabla_y f(y) = J_y(x)^{-T} \nabla_x f(x). \tag{19}$$

Integrals using Clenshaw–Curtis quadrature can also be performed on nonlinear triangles. This is done by transforming the integral back to the x domain and performing the integration there. For example,

$$\int_{\Omega_y} F(f(y), \nabla_y f(y)) \mathrm{d}y = \int_{\Omega_x} |J_y(x)| F(f(\phi(x)), J_y(x)^{-T} \nabla_y f(\phi(x))) \mathrm{d}x \tag{20}$$

where Ω_y is the nonlinear triangle and Ω_x is the equilateral triangle (see Figure 6). This integral can then be evaluated by Clenshaw–Curtis quadrature and numerical gradient computations.

The computational cost of computing the Jacobian of the map at each point in the domain is approximately twice the cost of computing a gradient. However, for a fixed mapping the Jacobian can be precomputed. The further computational costs associated with the above modifications to the gradient and Clenshaw–Curtis quadrature are also $\mathscr{O}(N^2)$ and only marginally increase the running time of the algorithms.

5.3 Linear transformations

If the mapping ϕ happens to be a linear mapping, then the gradient and Clenshaw–Curtis quadrature techniques for the transformed triangles simplify as the Jacobian $J_y(x)$ is a constant matrix for such transformations.

Thus, when calculating the gradient, the inverse transpose of the Jacobian need only be calculated once, however, it must still be applied to each point in the domain. Furthermore, in Clenshaw–Curtis quadrature, the constant factor of $|J_y(x)|$ in the integral of $P_N(y)$ can be applied after the integral of $P_N(x)$ is calculated. That is, Eq. (20) becomes

$$\int_{\Omega_y} F(f(y), \nabla_y f(y))\mathrm{d}y = |J_y(x)| \int_{\Omega_x} F(f(\phi(x)), J_y(x)^{-T} \nabla_y f(\phi(x)))\mathrm{d}x \tag{21}$$

These simplifications provide a marginal improvement in the running times of the gradient and Clenshaw–Curtis algorithms.

6 Numerics

In this section, we show timing and convergence results for the gradient and Clenshaw–Curtis quadrature algorithms using the test function $\exp(\sin(y_1)\sin(y_2))$ on the triangle with corners $\{(0,0),(0,1),(1,0)\}$. We also show timing and convergence results for the calculation of the surface area of a spherical triangle, which requires the use of both algorithms on a non-linearly transformed triangle. All computations are performed using a combination of MEX and MATLAB R2007b on a single 2.4GHz processor.

Quadratic reference timing curves (fitted using least squares) have also been plotted to emphasise the efficiency of the algorithms. Again, we would like to emphasise that there are $\mathscr{O}(N^2)$ points within the fundamental domain, of which, just over half lie within the triangle, thus our methods achieve spectral accuracy with nearly linear computational complexity in the number of sample points.

We begin with Clenshaw–Curtis quadrature of our test function on the triangle. Figure 7 shows the spectral rate of convergence of the Clenshaw–Curtis quadrature algorithm for a sufficiently smooth function. Most of the variation in the timing curve comes from the two dimensional fast Fourier transform (which is performed by FFTW) and is reproducible (cf. Figures 8–9).

If instead of the test function $\exp(\sin(y_1)\sin(y_2))$, one uses a monomial of degree p, one finds that the Clenshaw–Curtis quadrature algorithm becomes exact for N greater than some value (typically around $N/2$). This occurs because the non-zero Fourier coefficients of the approximation $P_N(x)$ for a monomial of degree p are limited to a hexagon of radius p centred at the origin of the dual fundamental domain. The required value of N to make the integration exact is then the smallest value of N such that the dual fundamental domain contains all of the k such that both $a(k) \neq 0$ and $\int_{\Omega_\theta} T_k(\theta)|J_x(\theta)|\mathrm{d}\theta \neq 0$. A similar result holds true for the gradient algorithm.

We now proceed to the calculation of the gradient of our test function using the recursion formula of Section 3. Figure 8 shows that the gradient algorithm also has a spectral rate of convergence. The two curves on the left of this figure are the L_2 norms of the absolute error in the y_1 and y_2 components of the gradient. The numerical error in the gradient for large N is entirely due to accumulated round-off error and grows as $\mathscr{O}(\varepsilon N^2)$, where ε is machine precision. This phenomenon is consistent with the general rule that numerical integration is stable while numerical differentiation is unstable.

Lastly, in Figure 9, we show convergence and timing results for our gradient and Clenshaw–Curtis quadrature algorithms applied to a nonlinear triangle. Instead of showing these separately, we demonstrate their use by calculating the surface area of the spherical triangle on the unit sphere with corners $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$, where $\hat{x}_i$ are normalised versions of the x_i:

$$x_1 = \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{\sqrt{3}} \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{\sqrt{3}} \\ 1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} \frac{2}{3} \\ 0 \\ 1 \end{bmatrix}.$$

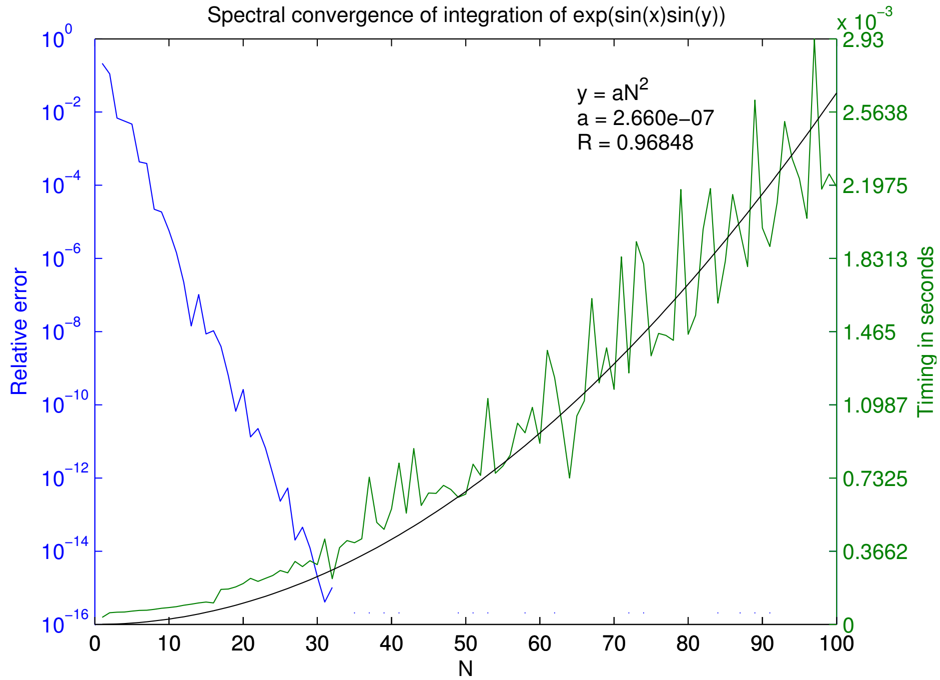


Fig. 7 Integration of $\exp(\sin(y_1)\sin(y_2))$ on the triangle.

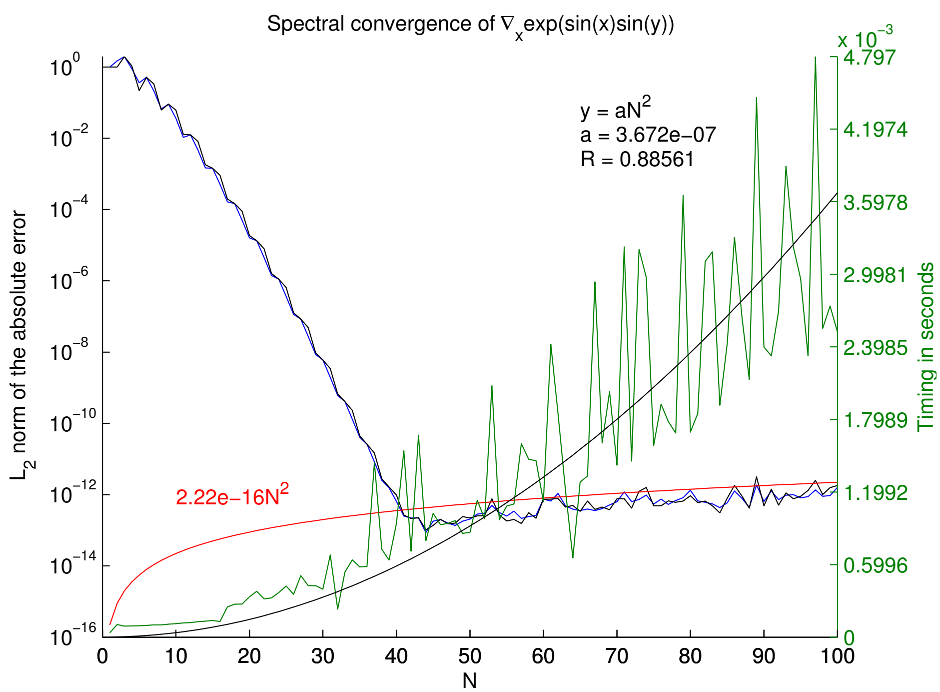


Fig. 8 Gradient of $\exp(\sin(y_1)\sin(y_2))$ on the triangle.

The surface area of a function $f(y_1,y_2)$ is calculated by first computing the gradient $\nabla_y f(y)$ and then by integrating over the domain the function

$$S = \sqrt{1 + \left(\frac{\partial f}{\partial y_1}\right)^2 + \left(\frac{\partial f}{\partial y_2}\right)^2}.$$

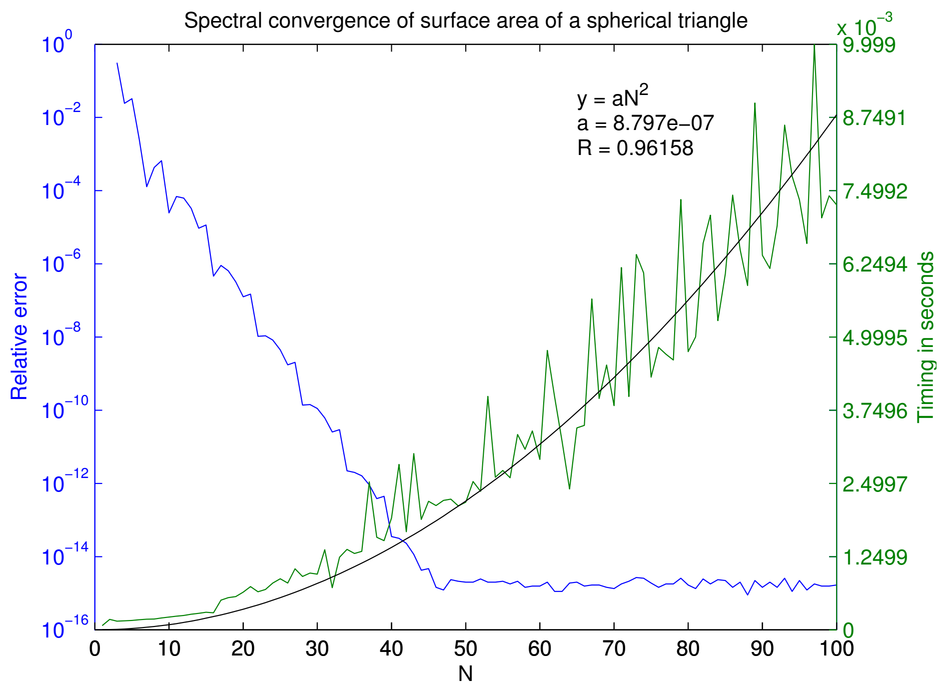


Fig. 9 Surface area of a spherical triangle with corners $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$.

From Figure 9 one can clearly see that the gradient and Clenshaw–Curtis quadrature algorithms perform as well and almost as fast for nonlinear maps as they do for linear maps.

7 Summary

In this paper, we have constructed a family of multivariate Chebyshev polynomials based on a symmetric extension of the fundamental domain of the affine Weyl group associated with a root system. Based on these multivariate Chebyshev polynomials, we have developed algorithms to approximate the gradient and the integral of functions over the fundamental domain associated with a root system. These algorithms are spectrally accurate and extremely fast, with computational complexity dominated by FFTs.

Here, we have focussed our attention on the A_2 root system, which is the simplest of the two-dimensional root systems with a triangular fundamental domain. This root system gives rise to multivariate Chebyshev polynomials that live on the interior of the deltoid $3(x_1^2+x_2^2)^2 - 8x_1(x_1^2-3x_2^2) + 6(x_1^2+x_2^2) = 1$. However, by restricting the domain of integration to the equilateral triangle that is inscribed within this deltoid, the Clenshaw–Curtis quadrature algorithm can be used to integrate over this triangle. Furthermore, given a (possibly nonlinear) mapping of this equilateral triangle, the Jacobian of this mapping can be computed numerically using the gradient algorithm, which allows for the computation of gradients and integrals on arbitrary (possibly nonlinear) triangles using our gradient and Clenshaw–Curtis quadrature algorithms. We have created MATLAB and C++ libraries of our algorithms for the A_2 root system, which will be made available at *http://hans.munthe-kaas.no/Chebyshev* for public use.

References

1. R. J. Beerends. Chebyshev polynomials in several variables and the radial part of the Laplace–Beltrami operator. *Transactions of the AMS*, 328(2):779–814, 1991.
2. L. P. Bos. Bounding the Lebegue function for lagrange interpolation in a simplex. *J. of Approx. Theory*, 38:43–59, 1983.
3. Q. Chen and I. Babuska. Aprroximate optimal points for polynomial interpolation of real functions in an interval and a triangle. *Comp. Meth. in App. Mech. and Eng.*, 128(3-4):405–417, 1995.
4. M. Dubiner. Spectral methods on triangles and other domains. *Journal of Scientific Computing*, 6(4):345–390, 1991.
5. R. Eier and R. Lidl. A class of orthogonal polynomials in k variables. *Math. Ann.*, 260:93–99, 1982.
6. A. F. Fässler and E. Stiefel. *Group theoretical methods and their applications*. Birkhäuser, Boston, 1992.
7. F. X. Giraldo and T. Warburton. A nodal triangle-based specral element method for the shallow water equations on the sphere. *J. of Comp. Physics*, 2005.
8. W. Heinrichs. Improved Lebesgue constant on the triangle. *J. of Comp. Physics*, 207:625–638, 2005.
9. J. S. Hesthaven. From electrostatics to almost optimal nodal sets for polynomial interpolation in a simplex. *SIAM J. Numer. Anal.*, 35(2):655–676, April 1998.
10. M. E. Hoffman and W. D. Withers. Generalized Chebyshev polynomials associated with affine Weyl groups. *Transactions of the AMS*, 308(1):91–104, 1988.
11. J. E. Humphreys. *Introduction to Lie algebras and representation theory*. Springer, 1970.
12. T. Koornwinder. Orthogonal polynomials in two variables which are eigenfunctions of two algebraically independent partial differential operators I–IV. *Indiag. Math.*, 36:48–66 and 357–381, 1974.
13. T. Koornwinder. Two-variable analogues of the classical orthogonal polynomials. In R. Askey, editor, *Theory and application of special functions*. Acad. Press, New York, 1975.
14. H. Li, J. Sun, and Y. Xu. Discrete Fourier analysis, cubature and interpolation on a hexagon and a triangle. *SIAM J. Numer. Anal*, 46:1653–1681, 2008.
15. R. Lidl. Tchebyscheffpolynome in mehreren Variabelen. *J. Reine Angew. Math*, 273:178–198, 1975.
16. H. Munthe-Kaas. Symmetric FFTs; a general approach. In *Topics in linear algebra for vector- and parallel computers, PhD thesis*. NTNU, Trondheim, Norway, 1989. Available at: `http://hans.munthe-kaas.no`.
17. H.Z. Munthe-Kaas. On group Fourier analysis and symmetry preserving discretizations of PDEs. *Journal of Physics A-Mathematical and General*, 39(19):5563–5584, 2006.
18. R. Pasquetti and F. Rapetti. Spectral element methods on triangles and quadrilaterals: comparisons and applications. *J. of Comp. Physics*, 198(1):349–362, 2004.
19. M. Puschel and M. Rötteler. Cooley-Tukey FFT like algorithm for the Discrete Triangle Transform. In *Proc. 11th IEEE DSP Workshop*, 2004.
20. M. A. Taylor, B. A. Wingate, and R. E. Vincnent. An Algorithm for Computing Fekete point in the triangle. *SIAM J. Numer. Anal.*, 38(5):1707–1720, 2000.
21. L.N. Trefethen. Is Gauss quadrature better than Clenshaw-Curtis? *Siam Review*, 50(1):67, 2008.
22. T. Warburton. An explicit construction of interpolation nodes on the simplex. *Journal of Engineering Mathematics*, 56(3):247–262, 2006.