ON SOME COMPUTATIONAL ASPECTS OF EQUILIBRIUM BUSINESS CYCLE THEORY*

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This paper evaluates the accuracy of quadratic approximate methods in the context of a simple real business cycle model. The level of accuracy is found to be very high. On the face of the results reported here we are confident that the different computation methods proposed to date in this literature can be employed interchangeably.

1. Introduction

It has been common practice in the recent Real Business Cycle (RBC) literature to approximate, quadratically, the return function about the steady state and then to use this approximate return function as the basis for generating the economy's equilibrium time series. This is done for well-known reasons of analytic and computational simplicity: with a quadratic return function the decision rules are linear and may be easily determined. To compute the optimal decision rules numerically via standard value iteration procedures is simply too intensive when the number of decision and state variables is large. Nevertheless, it is legitimate to question the extent to which accuracy is compromised using such approximate procedures.

The answer to this question may very well be entirely model-specific. Indeed, if it were feasible to undertake such a comparative evaluation for all model contexts, there would be no need to resort to such approximations in

the first place. In this paper we undertake such a comparison for the basic one-good stochastic growth model as analyzed in Hansen (1985) (his 'divisible labor economy'). That is, we solve for the optimal decision rules and equilibrium time series using both standard value iterative and quadratic approximation techniques and compare results. Some familiarity with the basic methodology is assumed.

2. The basic model

The model we investigate is a familiar one:

$$\max \mathrm{E}\left(\sum_{t=0}^{\infty} \beta^t u(c_t, 1 - N_t) \right),\tag{1}$$

subject to

$$c_t + z_t \le f(k_t, N_t)\tilde{\lambda}_t, \qquad 0 \le N_t \le 1,$$

$$k_{t+1} = z_t + (1 - \Omega)k_t, \qquad k_0 \text{ given},$$

where c_t, k_t, N_t, and z_t are, respectively, per capita consumption, capital stock, labor service supplied, and investment in period t, E is the expectations operator, β the period discount factor, Ω the depreciation factor, $u(\cdot)$ the representative agent's period utility function, $f(\cdot)$ the production function, and $\tilde{\lambda}_t$ the random period shock to technology. The precise functional forms employed here are the two preference orderings

$$u(c_t, 1 - N_t) = \ln(c_t) + 2\ln(1 - N_t),$$

$$u(c_t, 1 - N_t) = \frac{1}{\delta}\left[c_t^{\gamma}(1 - N_t)^{1-\gamma} \right]^{\delta},$$

in conjunction with the technology

$$f(k_t, N_t) = \frac{L}{\nu}\left[k_t^{\alpha} N_t^{1-\alpha} \right]^{\nu}.$$

For $\nu = L = 1$ and $\alpha = 0.36$, the above technology specification is the same as Hansen (1985); the preference specifications are taken from, respectively, Hansen (1985) and Kydland and Prescott (1982).

The choice of shock process requires a somewhat more extensive discussion. In Hansen's (1985) work, the shock to technology is described by a first-order Markov process of the form

$$\lambda_{t+1} = \rho\lambda_t + \tilde{\varepsilon}_t,\tag{2}$$

with $\rho = 0.95$ and $\tilde{\varepsilon}_t$ lognormally distributed with $E\tilde{\varepsilon}_t = 1 - \rho = 0.05$ and $\sigma_\varepsilon = 0.00712$.[1] While such a shock process is perfectly natural for the quadratic return, linear decision rule setting, it does not immediately translate into the standard value iterative context in which such approximations are not employed. The reason for this is as follows: under the standard approach, it is first necessary to specify, precisely, the region in which the state variables will assume their values over the infinite horizon. For the problem under discussion, the state variables are capital stock and the shock to technology. With regard to the range of the technology shock, it is customary in much of this literature [see, e.g., Danthine and Donaldson (1981), Mehra and Prescott (1985), and Greenwood and Huffman (1986)] to enumerate a discrete set of M possible values of λ, the probability of relative occurrence of which is governed by a prespecified M-dimensional probability transition matrix. For Hansen's (1985) technology choice, the range of the feasible capital stock levels will then be bounded above by some k_{max} defined by $f(k_{max}, 1)\bar{\lambda} = k_{max}$, where $\bar{\lambda}$ is the maximum possible shock value. Thus, the state variables are known to assume values only in the domain $[0, k_{max}] \times \{\lambda_1, \dots, \lambda_M\}$. Since this set is compact, the sequence of approximating optimization problems (value iteration) is well defined.

Compactness is lost under specification (2), however. In particular, Hansen's (1985) shock process is not uniformly bounded above over the infinite horizon and thus has no immediate representation in the transition matrix setting. This observation forces us to choose between these alternative shock structures. That is, either we choose an autoregressive process as per (2) – necessitating quadratic approximate solution techniques – or a discrete Markov process admitting standard value iterative procedures. In order to undertake our comparisons, we choose to specify a two-state Markov process on the shock to technology. Thus, for *both* the standard and quadratic approximate procedures we assume that the Markov process on the technology shock assumes values $\lambda_t \in \{\underline{\lambda}, \bar{\lambda}\}, \underline{\lambda} < \bar{\lambda}$, with transition density

$$\lambda_{t+1} = \quad \underline{\lambda} \qquad \bar{\lambda},$$

$$\lambda_t = \frac{\lambda}{\bar{\lambda}}\begin{bmatrix} \pi & 1 - \pi \\ 1 - \pi & \pi \end{bmatrix}.$$

Here π and $1 - \pi$ denote the respective period-to-period transition probabilities. In what follows, the conditional distribution of next period's shock given today's shock of λ will be represented as $dF(\lambda_t; \lambda)$.

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[1]Kydland and Prescott (1982) employ a more complex shock structure. It remains closely related to the one presented here. See section 7.

We note that for $\underline{\lambda} = 0.98$, $\bar{\lambda} = 1.02$, and $\pi = 0.97$, the above process will have the same mean and variance as the process chosen by Hansen (1985).[2]

3. The standard solution

3.1. Overall procedure

The sequence of approximating value functions is described by the recursive equation

$$V_n(k, \lambda) = \max_{\{k_n, N_n\} \in \Gamma} \left\{ u\left[f(k, N_n)\lambda - k_n + (1 - \Omega)k, 1 - N_n \right] \right.$$

$$\left. + \beta \sum_{t=1}^{2} V_{n-1}(k_n, \lambda_t) \, dF(\lambda_t; \lambda) \right\}, \tag{3}$$

subject to

$$(1 - \Omega)k \le k_n \le f(k, N_n)\lambda + (1 - \Omega)k, \qquad 0 \le N_n \le 1,$$

where the subscript n denotes the nth iteration and Γ is the domain of the choice variables k_n and N_n. Certain aspects of this formulation merit discussion.

3.1.1. The choice of 'grid'

The state and decision variables are constrained to assume values in a discrete set or 'grid' which reasonably approximates their actual domain of definition. We denote this set by Γ. This Γ represents a partition of $[0, k_{max}] \times [0, 1]$. The fineness or 'norm' of the grid – the maximum distance between successive grid partition elements – was itself an object of choice. Furthermore, different degrees of fineness could be chosen for the grid defining the domain of definition of k_n and the grid defining the domain of N_n. In general, the time-series statistics seemed less sensitive to the fineness of the N_n grid. The choice of the grid norm reported in the tables to follow was such that finer partitions, which increased computational time, did not alter the reported statistics in the first and second decimal places.

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[2] This was accomplished using the method of moments.

3.1.2. The choice of optimizing variables

Note that in the above formulation we optimize over next period's capital stock rather than investment as would be the case in the following:

$$
V_n(k, \lambda) = \max_{\{z_n, N_n\} \in \Gamma} \left\{ u\big(f(k, N_n)\lambda - z_n, 1 - N_n \big) \right.
$$

$$
\left. + \beta \sum_{t=1}^{2} V_{n-1}\big((1 - \Omega)k + z_n, \lambda_t\big) \, \mathrm{d}F(\lambda_t; \lambda) \right\}, \qquad (4)
$$

subject to

$$
0 \le z_n \le f(k, N_n)\lambda, \qquad 0 \le N_n \le 1.
$$

By optimizing directly over next period's capital [as per (3)], we eliminate round-off problems that might occur if next period's capital choice is not a grid point: indeed, k_n is chosen directly from the set of feasible values and investment z and consumption c are computed as residuals. If, as per (4), we optimize over z, there is no assurance that $(1 - \Omega)k + z_n$ will be a grid point. In all likelihood, it will have to be discretized (rounded up or down to the nearest grid value) and depending on how this is done, the results of the recursive computation can be very materially affected. We have found that optimizing over z is generally less precise than optimizing over k.

3.1.3. The domain of optimization

A simple serial search procedure for determining the optimal (k_n^*, N_n^*) by evaluating the above expression for all possible $(k_n, N_n) \in \Gamma$ is computation-ally very slow.[3] Although this number of calculations can be substantially reduced by applying known theoretical properties of the decision variables (such as the fact that k_n is increasing in k and λ), in light of the hundred plus iterations that may be necessary for policy convergence, the total computation time is substantial. The constraint requirement that $k_n \ge (1 - \Omega)k$ is one simple device for reducing the calculations. Since we know from theoretical analyses that $V_n(\cdot)$ is increasing in k for any n, it will never be optimal to dispose of capital, and thus the solution to the recursive optimization will never be affected by the imposition of this constraint.

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[3]Assuming as many as 250 distinct possible values for the capital stock, two values of the technology shock and 100 possible levels of labor supply, each stage of the iterative process would require $(250 \times 2) \times (250 \times 100)$ (number of possible state variable combinations $\times$ number of possible pairs of decision levels) $= 12{,}500{,}000$ independent evaluations of the above expression to determine the maximum.

Another device for reducing these computations very substantially is to devise a simple and rapid procedure for determining directly the optimal N_n associated with each possible choice of k_n. To see why this is so, we observe that for any (k, λ) and k_n the optimal associated N_n^* can be computed by maximizing

$$u\big(f(k, N)\lambda - k_n + (1 - \Omega)k, 1 - N\big)$$

alone with respect to N as N does not appear as an argument of the $V_{n-1}(\cdot)$ function. This is a fairly straightforward exercise that can be designed to execute very rapidly. We employ this technique in our algorithm; a precise description of how it is done is found in appendix 1. The impact of this subroutine is to reduce, effectively, the number of evaluations by a factor equal to the number of grid points in the domain of N_n.

3.1.4. Convergence criterion

The recursive routine defined by (3) must be terminated at some point and an important issue is to decide at what stage this should occur. We chose to terminate the routine whenever, simultaneously,

$$\sup_{(k, \lambda) \in \Gamma} |V_n(k, \lambda) - V_{n-1}(k, \lambda)| < c\|K\|,$$

$$\sup_{(k, \lambda) \in \Gamma} |N_n(k, \lambda) - N_{n-1}(k, \lambda)| \leq c\|K\|,$$

and

$$\sup_{(k, \lambda) \in \Gamma} |k_n(k, \lambda) - k_{n-1}(k, \lambda)| \leq c\|K\|,$$

where c is a constant of choice and $\|K\|$ denotes the norm of the capital stock partition. In the work to follow, $c = 0.5$. This latter value was chosen as a more restrictive criterion ($c < 0.5$) did not appear to materially alter the time-series statistics while substantially increasing the computation time.

3.1.5. Time-series properties

Let n^* denote the final iteration in the recursive routine; $N_{n^*}(k, \lambda)$ and $k_{n^*}(k, \lambda)$ thus represent our proxies for the optimal policy functions. The time-series output of the model was constructed by first generating a sequence of three thousand shocks to the technology with respect to the chosen transition matrix and then by allowing the economy to evolve from an arbitrarily determined initial state via repeated application of these optimal policy rules.

More specifically, let $\{\hat{\lambda}_t\}_{t=0}^{t=2999}$ denote the generated shock sequence. The corresponding capital stock sequence $\{k_t\}_{t=0}^{t=2999}$ was constructed via k_0 given and $k_{t+1} = k_n^*(k_t, \hat{\lambda}_t)$. The corresponding labor (or 'hours') series $\{N_t\}_{t=0}^{t=2999}$ was computed directly using $N_t = N_n^*(k_t, \hat{\lambda}_t)$, while the output series $\{Y_t\}_{t=0}^{t=2999}$ was defined by $Y_t = f(k_t, N_t)\hat{\lambda}_t$. Lastly, the consumption $\{c_t\}_{t=0}^{t=2999}$ and investment $\{z_t\}_{t=0}^{t=2999}$ series, respectively, were defined according to $c_t = Y_t - k_n^*(k_t, \hat{\lambda}_t) + (1 - \Omega)k_t$, and $z_t = k_n^*(k_t, \hat{\lambda}_t) - (1 - \Omega)k_t$. Output per hours worked $\{w_t\}_{t=0}^{t=2999}$ – average productivity – followed from the definition $w_t = Y_t/N_t$. It should also be noted that every time series was generated using exactly the same shock sequence. Means and variances were computed directly from the time series themselves (the data was not logged or otherwise transformed first).

In order to be assured that the economy had entered the support of the steady-state capital stock distribution, the first one thousand elements were dropped from all series. The decision to compute statistics on the basis of two thousand data points was empirically based: the use of more data points only altered the values of the computed statistics in their fourth decimal place.[4] We have not subjected the data to any detrending procedure as we thought it worthwhile first to assess the model's direct output without the modifications a smoothing procedure would introduce.[5]

3.1.6. Model integrity check

For our choice of model, it is a simple matter to compute the steady-state certainty ($\lambda_t \equiv 1$) levels of capital stock and employment. Specializing the steady-state formulations of Kydland and Prescott (1982) to our simplified setting yields the steady-state expressions

$$k^{ss} = \left[\frac{1}{\alpha} \left(\frac{1}{\beta} - 1 + \Omega \right) \right]^{1/(\alpha - 1)} \cdot N^{ss},$$

where

$$N^{ss} = (1 - \alpha) \left(\frac{1}{\beta} - 1 + \Omega \right) \left(3 \left[\frac{1}{\beta} - 1 + \Omega \right] - \alpha \left[\frac{1}{\beta} - 1 + 3\Omega \right] \right)^{-1}.$$

A natural check on the program's integrity is to see if the recursive routine and time-series generation gives these steady-state levels when $\underline{\lambda} = \bar{\lambda} = 1$. For all

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[4]It is also possible to compute the transition matrix on capital stock–shock pairs and from this compute the stationary distribution via matrix manipulation. We have found this to be more computationally intensive than the method employed here at no appreciable gain in accuracy.

[5]Since the output data of this model does not, by construction, exhibit any trend, it is not entirely clear that such a procedure would be appropriate anyway.

Table 1

$\beta = 0.96$, $\alpha = 0.36$, $\gamma = 0.33$, $\Omega = 0.1$, $\bar{\lambda} = 1.02$, $\underline{\lambda} = 0.98$, $\delta = -1$, $\pi = 0.97$, $L = 1$.

Norm of capital stock partition = 0.0025 (401 capital stock grid points on [0.5, 1.5]).

Norm of labor partition = 0.0025 (401 labor grid points on [0.0025, 1.0025]).

Technology: $f(k_t, N_t) = Lk_t^\alpha N^{1-\alpha}$.

Preferences: Case A: $u(c_t, 1 - N_t) = (1/\delta)[c_t^\gamma(1 - N_t)^{1-\gamma}]^\delta$,

Case B: $u(c_t, 1 - N_t) = \ln c_t + 2\ln(1 - N_t)$.

	Case A[a]		Case B[a]	
	(a)	(b)	(a)	(b)
Output	2.98	1.00	3.06	1.00
Consumption	2.74	0.983	2.76	0.973
Investment	4.13	0.930	4.56	0.909
Capital stock	2.78	0.864	2.92	0.870
Hours	0.41	0.500	0.52	0.495
Productivity (average)	2.79	0.991	2.84	0.986

[a](a) standard deviation in percent, (b) correlation with output.

routines considered here, this was the case to at least two decimal points accuracy.

We performed the above mentioned simulations for a wide variety of parameter values. Here we report the results of the exercise for Hansen's (1985) preferences and technology (case B) and parameter values conforming to those of Kydland and Prescott (1982) with, once again, Hansen's (1985) technology parameter values (case A).[6] Limiting ourselves to the summary statistics reported in Hansen (1985), we find the time-series properties of this optimal economy summarized in table 1.

3.2. Numerical results

We note that for a state space of 401 possible capital stock levels and 401 possible labor supply levels, the entire computational procedure (each case above) required approximately 85 minutes of VAX-11-780 CPU time.

4. A quadratic approximate solution

4.1. Numerical procedure

Our procedure is similar to that of section 3 except that we undertake a quadratic approximation of the return function about the economy's certainty

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[6] We chose to include the more general preference structure of Kydland and Prescott (1982) as it afforded the option of testing the sensitivity of results vis-a-vis preference parameters.

steady-state. As before, we first outline our iterative procedure and then present our comparative time-series results.

Let the superscript ss denote the steady-state certainty ($\lambda_t \equiv 1$) value of the relevant variable.[7] Following Kydland and Prescott (1982) and Hansen (1985), we approximate the agent's utility function about the steady-state by the following expression:

$$U^*(k, N, \lambda, z) = \overline{U} + BX + X'QX, \tag{5}$$

where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} k - k^{ss} \\ N - N^{ss} \\ \lambda - \lambda^{ss} \\ z - z^{ss} \end{bmatrix}, \tag{6}$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \begin{bmatrix} [\hat{u}(k^{ss} + \Delta_k, N^{ss}, \lambda^{ss}, z^{ss}) - \hat{u}(k^{ss} - \Delta_k, N^{ss}, \lambda^{ss}, z^{ss})]\dfrac{1}{2\Delta_k} \\[8pt] [\hat{u}(k^{ss}, N^{ss} + \Delta_N, \lambda^{ss}, z^{ss}) - \hat{u}(k^{ss}, N^{ss} - \Delta_N, \lambda^{ss}, z^{ss})]\dfrac{1}{2\Delta_N} \\[8pt] [\hat{u}(k^{ss}, N^{ss}, \lambda^{ss} + \Delta_\lambda, z^{ss}) - \hat{u}(k^{ss}, N^{ss}, \lambda^{ss} - \Delta_\lambda, z^{ss})]\dfrac{1}{2\Delta_\lambda} \\[8pt] [\hat{u}(k^{ss}, N^{ss}, \lambda^{ss}, z^{ss} + \Delta_z) - \hat{u}(k^{ss}, N^{ss}, \lambda^{ss}, z^{ss} - \Delta_z)]\dfrac{1}{2\Delta_z} \end{bmatrix}, \tag{7}$$

where

$$\hat{u}(k, N, \lambda, z) = u(f(k, N)\lambda - z, 1 - N),$$

$$\overline{U} = \hat{u}(k^{ss}, N^{ss}, \lambda^{ss}, z^{ss}), \tag{8}$$

and Q is a 4×4 matrix with diagonal entries of the form

$$q_{11} = [\hat{u}(k^{ss} + \Delta_k, N^{ss}, \lambda^{ss}, z^{ss}) - 2\hat{u}(k^{ss}, N^{ss}, \lambda^{ss}, z^{ss})$$

$$+ \hat{u}(k^{ss} - \Delta_k, N^{ss}, \lambda^{ss}, z^{ss})]\frac{1}{2\Delta_k^2},$$

$$\vdots \tag{9}$$

$$q_{44} = [\hat{u}(k^{ss}, N^{ss}, \lambda^{ss}, z^{ss} + \Delta_z) - 2\hat{u}(k^{ss}, N^{ss}, \lambda^{ss}, z^{ss})$$

$$+ \hat{u}(k^{ss}, N^{ss}, \lambda^{ss}, z^{ss} - \Delta_z)]\frac{1}{2\Delta_k^2},$$

[7]These steady-state values were computed as in Kydland and Prescott (1982).

and off-diagonal elements of the form, e.g.,

$$q_{13} = \left[\hat{u}\left(k^{ss} + \Delta_k, N^{ss}, \lambda^{ss} + \Delta_\lambda, z^{ss} \right) - \hat{u}\left(k^{ss} - \Delta_k, N^{ss}, \lambda^{ss} + \Delta_\lambda, z^{ss} \right) \right.$$

$$- \hat{u}\left(k^{ss} + \Delta_k, N^{ss}, \lambda^{ss} - \Delta_\lambda, z^{ss} \right)$$

$$\left. + \hat{u}\left(k^{ss} - \Delta_k, N^{ss}, \lambda^{ss} - \Delta_\lambda, z^{ss} \right) \right] \frac{1}{(\Delta_k \Delta_\lambda)^8}. \tag{10}$$

In the above expressions, the symbol Δ indicates the deviation from the steady-state, while $\bar{U}$ represents the certainty steady-state level of period utility. The deviations themselves were a parameter of choice.

Turning to the iterative procedure itself, it is evident that for fixed (k, λ), the associated period-zero value function $V_0^Q(k, \lambda)$ can be expressed as

$$V_0^Q(k, \lambda) = \text{Constant} + b_2 x_2 + \sum (q_{2j} + q_{j2}) x_j x_2 + q_{22} x_2^2, \tag{11}$$

where the constant has the value

$$\text{Constant} = \bar{u} + \sum_{i \neq 2} b_i x_i + \sum_{\substack{i \neq 2 \\ j \neq 2}} q_{ij} x_i x_j$$

and $\tag{12}$

$$X_4 = (1 - \Omega)k - z^{ss}.$$

The superscript Q indicates that the value function employs the quadratic approximation. We note also that the precise representations of the x_i's, b_i's, and q_{ij}'s have been suppressed for ease of presentation.

Noting that eq. (11) is of the form $V = F + Ex_2 + Dx_2^2$, where D, E, and F are appropriate constants, it is clear that the maximum with respect to x_2, for fixed (k, λ), will occur at

$$x_2 = -\frac{E}{2D} = \frac{-\left(\sum_{j \neq 2} 2q_{j2} x_j + b_2 \right)}{2 q_{22}}, \tag{13}$$

from which the optimal $N_0 = N_0(k, \lambda) = N^{ss} + x_2$ is easily computed. It follows that

$$V_0^Q(k, \lambda) = u\left(f(k, N_0)\lambda, 1 - N_0 \right) \quad \text{for all} \quad (k, \lambda). \tag{14}$$

More generally,

$$
V_n^Q(k, \lambda) = \max_{\substack{\{k_n, N_n\} \in \Gamma \\ 0 < N_n < 1}} \left\{ U^*(k, N_n, \lambda, k_n) \right.
$$

$$
\left. + \beta \sum_{t=1}^{2} V_{n-1}^Q(k_n, \lambda_t) \, dF(\lambda_t; \lambda) \right\}, \tag{15}
$$

$$
(1 - \Omega)k \le k_n \le f(k, N_n)\lambda - (1 - \Omega)k.
$$

As in the routine of section 2, we optimize over next period's capital stock rather than this period's investment. This requires some modification of $U^*(\cdot)$, but again seems substantially to reduce the consequences resulting from rounding off.

The sequence of maximizations described by (15) was solved using a modified grid search procedure. The advantage of the quadratic approximation comes from the fact that at each stage of the recursion the optimal N_n can be exhibited directly without the need of any additional subroutine. This is detailed in appendix 2. The resulting savings policy function is linear and is the same as if this expression had also been solved directly.

4.2. Numerical results

For the results reported in table 2, we chose deviations proportional to the steady-state values with proportionality constant $\Delta = 0.00001$ (e.g., $\Delta_k = 0.00001k^{ss}$). All other parameters are the same as in table 1. We note that each of these cases required approximately 25 minutes of VAX-11-780 CPU time.

Comparing the corresponding entries in tables 1 and 2, it is clear that the standard deviations and correlations are virtually identical for both cases. Although the statistics are not reported, mention should also be made that the levels (expected values) of all the series were, respectively, essentially identical under the standard or quadratic approximate procedures. Thus – and this is the central message of the paper – one may conclude that the precision of the approximation afforded using quadratic approximate methods appears remarkable. Of course, one cannot eliminate the possibility that our simple context overstates this precision; nevertheless, we have displayed two examples for which both methods are feasible and yield essentially the same results.

One extra parameter that enters into the calculation when using quadratic approximate methods is the Δ. The calculations behind table 2 relied on equiproportionate deviations, though, in fact, we have no formal theory as to what these deviations should be. In the next section we examine the sensitivity of the approximation to varying choices of deviations.

Table 2

$\beta = 0.96$, $\alpha = 0.36$, $\gamma = 0.33$, $\Omega = 0.1$, $\bar{\lambda} = 1.02$, $\underline{\lambda} = 0.98$, $\delta = -1$, $\pi = 0.97$, $\Delta = 0.00001$, $L = 1$.

Norm of capital stock partition = 0.0025 (401 capital stock grid points on [0.5, 1.5]).

Norm of labor partition = 0.0025 (401 labor grid points on [0.0025, 1.0025]).

Technology: $f(k_t, N_t) = Lk_t^\alpha N^{1-\alpha}$.

Preferences: Case A: $u(c_t, 1 - N_t) = (1/\delta)[c_t^\gamma(1 - N_t)^{1-\gamma}]^\delta$,

Case B: $u(c_t, 1 - N_t) = \ln(c_t) + 2\ln(1 - N_t)$.

	Case A[a]		Case B[a]	
	(a)	(b)	(a)	(b)
Output	3.03	1.000	3.05	1.000
Consumption	2.74	0.982	2.77	0.973
Investment	4.33	0.934	4.58	0.905
Capital stock	2.95	0.868	2.93	0.874
Hours	0.41	0.528	0.50	0.467
Productivity (average)	2.83	0.993	2.86	0.988

[a](a) standard deviation in percent, (b) correlation with output.

5. Choice of deviations

Lacking a theory specifying the selection of deviations, any scheme must necessarily be somewhat ad hoc. In this section, we attempt to assess the sensitivity of the approximation's accuracy to the specification of deviations by replicating the exercise of tables (1) and (2) for a variety of other choices. Three alternatives were considered; they are detailed below:

5.1. Kydland Prescott (1982) deviations

Reasoning that our model is a special case of Kydland and Prescott's (1982), we evaluated the approximation under their specification: $\Delta_k = 0.0001k^{ss}$, $\Delta_N = 0.0003N^{ss}$, $\phi_\lambda = 0.0003\lambda^{ss}$, $\Delta_z = 0.0008z^{ss}$.

5.2. Proportional absolute deviations

In this model, with $\Omega = 0.10$, z^{ss} will have a value one tenth k^{ss}. As table 1 suggests, however, investment will vary proportionally much more than capital stock. The equiproportionate deviation scheme reported in table 1 may not adequately capture the relative dispersion of the various series. As an attempt to respond to this concern, we next evaluated the approximation for absolute deviations that were in the same ratio as the standard deviation per mean of the counterpart series under the standard solution. Note that this exercise is

for evaluative purposes only as there is no reason to consider the approximation if the standard solution is computationally feasible, as must be the case for this scheme to be implemented.

For the specification of, e.g., case A in table 1, this resulted in the deviations $\Delta_k = 0.0000278$, $\Delta_N = 0.0000041$, $\Delta_\lambda = 0.000020$ (for our shock process, the standard deviation/mean is 2.0), and $\Delta_z = 0.0000413$. Given that $Ek = 1.28$, $EN = 0.2957$, $Ez = 0.128$, and $E\lambda = 1.00$ (E denotes expected values), the above deviations are of the same order of magnitude as those employed in table 2.

5.3. Equal absolute deviations

For this final experiment we chose $\Delta_\lambda = \Delta_N = \Delta_z = \Delta_k = 0.00001$. The results of this exercise are reported in table 3 for the functional forms and parameter values of table 1. The message of this exercise must be that the quadratic approximation retains its accuracy across a wide class of deviations schemes, provided that the absolute magnitude of the deviations is very small – not larger than the order of magnitude 0.0001.[8] Note also that the discrepancies noted in table 3 fall within the confidence intervals proposed by users of quadratic approximate methods.

The one remaining evaluative issue concerns the robustness of these results to the choice of parameter values. This is considered in the next section.

6. Sensitivity analysis

In order to be assured that our results were not peculiar to the specific parameterizations already reported, we performed the exercise of tables 1, 2, and 3 for a wide range of parameter values. Restricting ourselves to the Kydland and Prescott (1982) preference specification, we considered various cases where $\beta \in \{0.5, 0.8, 0.96\}$, $\gamma \in \{0.33, 0.5, 0.75\}$, $\Omega \in \{0.5, 0.1, 1\}$, and $\delta \in \{-2, -1, 0.5\}$ in conjunction with the four deviation schemes detailed in sections 3 and 4.

A detailed reporting of the results of these many cases would be wearisome and unnecessary. Let it suffice to provide the following descriptive summary:

(i) With regard to relative volatility [column (a)] the accuracy of the approximation was of the same general level as reported in tables 2 and 3 for all sets of parameter values considered. The greatest discrepancies were observed with the investment series. For the majority of cases, the approximation was best when the deviations in some way accounted for the relative volatility of the different series.

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[8] For significantly larger deviations (e.g., 0.01) the approximation loses its accuracy.

Table 3

$\alpha = 0.36$, $\beta = 0.96$, $\gamma = 0.33$, $\Omega = 0.1$, $\bar{\lambda} = 1.02$, $\underline{\lambda} = 0.98$, $\delta = -1$, $\pi = 0.97$, $L = 1$.

Norm of capital stock partition = 0.0025 (401 capital stock grid points on [0.5, 1.5]).

Norm of labor partition = 0.0025 (401 labor grid points on [0.0025, 1.0025]).

Technology: $f(k_t, N_t) = Lk_t^\alpha N_t^{1-\alpha}$.

Preferences: Case A: $u(c_t, 1 - N_t) = (1/\delta)[c_t^\gamma (1 - N_t)^{1-\gamma}]^\delta$,

Case B: $u(c_t, 1 - N_t) = \ln(c_t) + 2\ln(1 - N_t)$.

(1) Kydland–Prescott deviations, (2) proportional absolute deviations,
(3) equal absolute deviations.

	Case A[a]					
	(1)		(2)		(3)	
	(a)	(b)	(a)	(b)	(a)	(b)
Output	3.01	1.000	3.05	1.000	2.97	1.000
Consumption	2.76	0.981	2.75	0.982	2.72	0.982
Investment	4.43	0.930	4.39	0.939	4.20	0.928
Capital stock	3.00	0.870	2.98	0.871	2.81	0.868
Hours	0.43	0.531	0.43	0.599	0.40	0.479
Productivity (average)	2.85	0.992	2.84	0.993	2.81	0.992

	Case B[a]					
	(1)		(2)		(3)	
	(a)	(b)	(a)	(b)	(a)	(b)
Output	3.01	1.000	3.06	1.000	3.01	1.000
Consumption	2.75	0.973	2.77	0.973	2.75	0.971
Investment	4.45	0.902	4.59	0.908	4.46	0.899
Capital stock	2.82	0.874	2.94	0.873	2.81	0.874
Hours	0.49	0.441	0.51	0.479	0.51	0.439
Productivity (average)	2.82	0.988	2.85	0.988	2.82	0.987

[a] (a) standard deviation in percent, (b) correlation with output.

(ii) In all cases, the relative volatility of the various series [column (a)] under the quadratic approximation (across all deviation schemes) was the same as under the standard solution; that is, consumption varied less than output, investment varied more than output, etc.

(iii) All correlations with output [column (b)] under the standard and quadratic approximate procedures were closely similar, with the greatest discrepancy observed with the hours series.

In summary, nothing was observed that would question the robustness of the approximation's accuracy to parameter changes.

Table 4

Continuous state-space methodology.

Technology: $f(k_t, N_t) = k_t^\alpha (1 - N_t)^{1-\alpha}$.

Preferences: $u(c_t, 1 - N_t) = \ln(c_t) + 2\ln(1 - N_t)$, $\lambda_{t+1} = \rho\lambda_t + \tilde{\varepsilon}_t$.

$\tilde{\varepsilon}_t \sim$ log normal, $E\tilde{\varepsilon}_t = 1 - \rho$, $\sigma_{\varepsilon_t} = 0.00712$.

$\alpha = 0.36$, $\beta = 0.96$, $\Omega = 0.1$, $\rho = 0.95$.

| | Hansen model[a] | |
	(a)	(b)
Output	3.02	1.0000
Consumption	2.70	0.973
Investment	4.53	0.920
Capital stock	3.02	0.867
Hours	0.50	0.544
Productivity (average)	2.78	0.988

[a](a) standard deviation in percent, (b) correlation with output.

7. Further considerations

Kydland and Prescott (1982) and Hansen (1985) employ a continuous state-space approach to model analysis: by making use of the quadratic properties, they derive the explicit linear decision rules appropriate to the (approximate) quadratic return function and then trace out the time path of the economy allowing the state variables to assume any value. Both computation routines employed in this paper, however, use a grid search procedure. In particular, the optimal investment function is only defined for a discrete set of values (the optimal N as a function of k and λ was solved for explicitly, however).[9] Employing essentially the same solution methodology across all comparisons enabled us thereby to isolate the precise effects of the approximation alone. The issue remains as to how well the Hansen (1985) and Kydland–Prescott (1982) methods and the methods considered in this paper correspond. This is especially important as, for large numbers of state/decision variables, the discrete state-space methods employed here become computationally prohibitive.

As an attempt to respond to this question, we compared the unfiltered output of Hansen's divisible labor continuous state-space model with that of our discrete analogue. Gary Hansen kindly provided the data in table 4. These

[9]The dramatic reduction in program execution time under the quadratic approximation can be attributed to the explicit solution of the optimal $N_t = N(k_t, \lambda_t)$ function and to various numerical simplifications the approximation allowed.

Table 5

Continuous state-space methodology.

Technology: $f(k_t, N_t) = k_t^\alpha (1 - N_t)^{1-\alpha}$.

Preferences: $u(c_t, 1 - N_t) = \ln(c_t) + 2\ln(1 - N_t)$.

Two-state shock process
$\alpha = 0.36, \beta = 0.96, \Omega = 0.1, \pi = 0.97, \bar{\lambda} = 1.02, \underline{\lambda} = 0.98$

	Two-state shock process[a]	
	(a)	(b)
Output	3.16	1.0000
Consumption	2.81	0.974
Investment	4.75	0.921
Capital stock	3.14	0.874
Hours	0.52	0.550
Productivity (average)	2.89	0.988

[a](a) = standard deviation in percent, (b) = correlation with output.

results are virtually identical to those of case B in tables 1 and 2. If the models underlying tables 1, case B, and 4 were identical in all respects, this coincidence would be sufficient to confirm the accuracy of the continuous state-space methods. The one difference, however, resides in the nature of the shock structure. [It is also the case that the data underlying table 4 has been logged, while tables 1 and 2 report statistics based on raw data. We assume this effect is minor and ignore it.]

To assess the impact of the different shock structures, we computed the time series and resultant statistics using Hansen's (1985) optimal decision rules (again provided by Gary Hansen in a private communication) in conjunction with our shock process. The idea was simply to see if the imposition of a two-state shock process would significantly alter the statistics reported in table 4. The results of this exercise are reported in table 5.

Tables 2 (case B), 4, and 5 together suggest that the decision rules underlying these reported statistics are relatively insensitive to the alternative shock structures considered in this paper per se. Indeed, the effect of imposing the two-state shock structure is to alter the statistics by less than 5% [comparing tables 4 and 5]. The fact that the statistics derived using continuous state-space decision rules [table 5] closely approximate the true statistics of table 1, case B, when the shock process is the same further confirms the legitimacy of the continuous state-space methodology.

As a final test of the continous state-space methods we directly compared the decision rules underlying tables 2 (case B) and 4. For the continous methodology and the parameter values and functional forms of table 4, the

resultant decision rules are

$$N_t = 0.23161 + 0.13237\lambda_t - 0.04946k_t, \tag{16}$$

$$z_t = -0.18884 + 0.40372\lambda_t - 0.06668k_t. \tag{17}$$

For the discrete case, no such rules were immediately available as, in particular, the optimal investment function was defined only on a finite set of (k, λ) pairs. To obtain expressions analogous to (16) and (17) above, we thus regressed the optimal $z = z(k, \lambda)$ and $N = N(k, \lambda)$ functions on k and λ for all (k, λ) in their domain of definition. The results of this exercise for the parameter values of table 2, case B, and equiproportionate deviations ($\Delta_x = 0.00001x^{ss}$ for $x = k, z, N, \lambda$) are reported below:

$$N_t = 0.2304 + 0.1290\lambda_t - 0.0493k_t, \tag{18}$$

$$z_t = -0.1900 + 0.3953\lambda_t - 0.0663k_t. \tag{19}$$

The variances of the residuals for (18) and (19) were, respectively, 0.00000136 and 0.00000582, which indicate a very good linear fit. Clearly, both methodologies generate essentially the same decision rules. In summary, both methods appear equally accurate with regard to decision rules derived and statistics thereby obtained.

8. Concluding comments

Our work suggests that quadratic approximate methods will provide a very good proxy for the conventional techniques whenever discrete state-space methods are used provided that the shocks are low and that the deviations which form the basis of the approximation are small. We have provided substantial evidence to support the accuracy of the continuous state-space quadratic methodology as well. On the face of our results, we are confident that these methods can be employed interchangeably as dictated by the feasibility requirements of each particular model context.

Appendix 1

We begin the detailed discussion of the procedure with the following observation:

(i) Searching over the region of ordered pairs (k_n, N_n) for which $0 \le N_n \le 1$ and $(1 - \Omega)k \le k_n \le f(k, N_n) + (1 - \Omega)k$ is equivalent to searching over

the region $(1 - \Omega)k \le k_n \le f(k, N_n) + (1 - \Omega)k$ and $N_0 \le N_n \le 1$, where

$$N_0 = N_0(k_n, k, \lambda) \equiv \left[\frac{\nu[k_n - (1 - \Omega)k]}{Lk^{\alpha\nu}\lambda} \right]^{1/(1-\alpha)\nu}.$$

(ii) Maximizing $U(\cdot) + V_{n-1}(\cdot)$ with respect to N_n is equivalent to maximizing $U(\cdot)$ alone with respect to N_n, since $V_{n-1}(\cdot)$ is independent of N_n.

We can solve for the optimal N_n associated with each (k_n, k, λ) triple by maximizing

$$\left(aN_n^{(1-\alpha)\nu} - b \right)^{\gamma\delta} (1 - N_n)^{(1-\gamma)\delta} c \tag{20}$$

with respect to N_n, where

$$a = a(k, \lambda) = Lk^{\alpha\nu}\lambda/\nu \ge 0,$$

$$b = b(k_n, k) = k_n - (1 - \Omega)k \ge 0,$$

$$c = 1/\delta.$$

Since eq. (20) is a concave function of N_n we may differentiate and set the resultant expression equal to zero to determine the optimal N_n. This yields, after some manipulation, the expression

$$\frac{\gamma a(1 - \alpha)\nu N_n^{(1-\alpha)\nu-1}}{aN_n^{(1-\alpha)\nu} - b} = (1 - \gamma)\frac{1}{1 - N_n}. \tag{21}$$

Rearranging terms this may be rewritten as

$$AN_n^{(1-\alpha)\nu} - BN_n^{(1-\alpha)\nu-1} - C = 0, \tag{22}$$

where

$$A = (1 - \gamma)a + \gamma a(1 - \alpha)\nu \ge 0,$$

$$B = \gamma a(1 - \alpha)\nu \ge 0,$$

$$C = (1 - \gamma)b.$$

Rearranging eq. (22) again gives

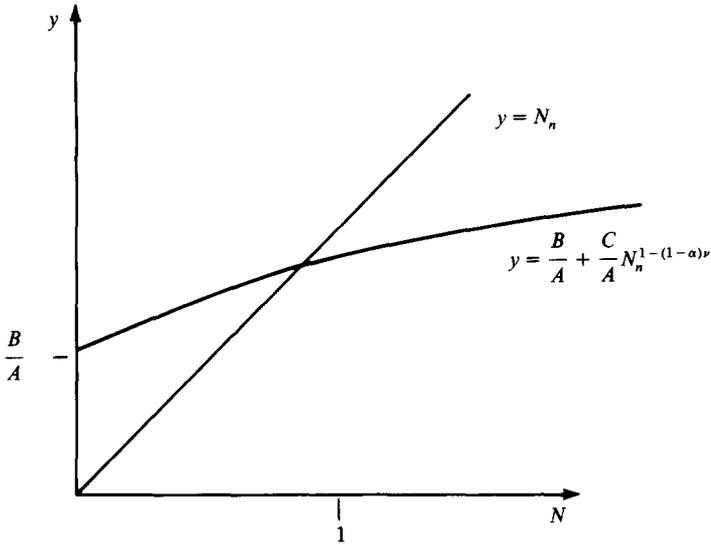$$AN_n - B - CN_n^{1-(1-\alpha)\nu} = 0 \tag{23}$$

Fig. 1

or

$$N_n = \frac{B}{A} + \frac{C}{A} N_n^{1-(1-\alpha)\nu}. \tag{24}$$

Eq. (24) is amendable to standard fixed-point procedures. Graphing each side of (24), as per fig. 1, it is apparent that the recursive iteration procedure, defined by

$$N_n^{(0)} \equiv 1,$$

$$N_n^{(1)} = \frac{B}{A} + \frac{C}{A}\left(N_n^{(0)}\right)^{1-(1-\alpha)\nu},$$

$$N_n^{(2)} = \frac{B}{A} + \frac{C}{A}\left(N_n^{(1)}\right)^{1-(1-\alpha)\nu}, \tag{25}$$

$$\vdots$$

$$N_n^{(k)} = \frac{B}{A} + \frac{C}{A}\left(N_n^{(k-1)}\right)^{1-(1-\alpha)\nu},$$

will approximate the solution. The remaining issue is to establish a conver-

gence criterion for the procedure. Suppose the solution to eq. (24) is $N_n = B/A + (C/A)N_n^{1-(1-\alpha)\nu}$. Then, for an $N_n^{(i)}$,

$$N_n^{(i+1)} = \frac{B}{A} + \frac{C}{A}\left(N_n^{(i)}\right)^{1-(1-\alpha)\nu}$$

$$= \frac{B}{A} + \frac{C}{A}(N_n)^{1-(1-\alpha)\nu}$$

$$+ \frac{C}{A}(1 - (1-\alpha)\nu)N_n^{-(1-\alpha)\nu}\left(N_n^{(i)} - N_n\right)$$

$$= N_n + \frac{C}{A}(1 - (1-\alpha)\nu)N_n^{-(1-\alpha)\nu}\left(N_n^{(i)} - N_n\right). \qquad (26)$$

If we desire a solution for which $N_n^{(i)} - N_n < 0.5\|K\|$, where $\|K\|$ denotes the width of the partition of possible capital stock levels, then the convergence requirement becomes

$$\left|N_n^{(i+1)} - N_n^{(i)}\right| \cong \left|N_n + \frac{C}{A}(1 - (1-\alpha)\nu)N_n^{-(1-\alpha)\nu}\left(N_n^{(i)} - N_n\right) - N_n^{(i)}\right|$$

$$= \left|\left[\frac{C}{A}(1 - (1-\alpha)\nu)N_n^{-(1-\alpha)\nu} - 1\right]\left(N_n^{(i)} - N_n\right)\right|$$

$$\leq \left|\frac{C}{A}(1 - (1-\alpha)\nu)N_n^{-(1-\alpha)\nu} - 1\right|(0.5\|K\|),$$

where C/A and N_n can be estimated from the steady-state certainty levels of capital and labor; that is,

$$\frac{C}{A} \cong \frac{(1-\gamma)[k^{ss} - (1-\Omega)k^{ss}]\nu}{[(1-\gamma) + \gamma(1-\alpha)\nu]Lk^{\alpha\nu}\lambda},$$

$$N_n \cong 0.3 = N^{ss},$$

and $(1 - (1-\gamma)\nu) = 0.36$ [in the case of Hansen's (1985) choice of parameters]. Given the determination of the optimal $N_n = N_n(k_n, k, \lambda)$, the related optimal k_n was determined via a modified exhaustive search procedure.

Appendix 2

An examination of the region defining the set of possible choices of the state variables allows us to express (15) equivalently as

$$
V_n^Q(k, \lambda) = \max_{k_n, N_n} \left\{ U^*(k, N_n, \lambda, k_n) \right.
$$

$$
\left. + \beta \sum_{t=1}^{2} V_{n-1}^Q(k_n, \lambda_t) \, dF(\lambda_t, \lambda) \right\}, \tag{27}
$$

subject to

$$
(1 - \Omega)k \le k_n \le f(k, N_n) + (1 - \Omega)k,
$$

$$
N_0(k_n, k, \lambda) \le N_n \le 1,
$$

where $N_0(k_n, k, \lambda)$ is defined as in appendix 1. Substituting and rearranging terms gives

$$
V_n^Q(k, \lambda) = \max_{(1-\Omega)k \le k_n \le f(k, N_n) + (1-\Omega)k} \left\{ \max_{N_0(k_n, k, \lambda) \le N_n \le 1} \left\{ C(k, \lambda, k_n) \right. \right.
$$

$$
\left. \left. + b_2 x_2 + \sum_{j \ne 2} (q_{j2} + q_{2j}) x_j x_2 + q_{22} x_2^2 \right\} \right\}, \tag{28}
$$

where

$$
C(k, \lambda, k_n) = \hat{u}(k^{ss}, N^{ss}, \lambda^{ss}, z^{ss}) + \sum_{i \ne 2} b_i x_i + \sum_{\substack{i \ne 2 \\ j \ne 2}} q_{ij} x_i x_j
$$

$$
+ \beta \sum_{t=1}^{2} V_{n-1}^A(k_n, \lambda_t) \, dF(\lambda_t, \lambda). \tag{29}
$$

As before the optimum N_n does not depend on $V_{n-1}^Q(\cdot)$ and can be computed directly by differentiating the expression of line (29). Since (29) is again of the form $V = F + EX_2 + DX_2^2$, this yields

$$
x_2 = -\frac{E}{2D} = \frac{-\left(b_2 + \sum_{j \ne 2} 2 q_{j2} x_j \right)}{2 q_{22}} \equiv N_n - N^{ss}. \tag{30}
$$

We note also that, if $N_n > 1$, we then define $N_n = 1$; if $N_n < N_0$, we define $N_n = N_0$. Given this information, the computation of the optimal k_n reduces to solving

$$V_n^Q(k, \lambda) = \max_{(1-\Omega)k \leq k_n \leq f(k,1)+(1-\Omega)k} \left\{ u^*(k, N_n, \lambda, k_n) \right.$$

$$\left. + \beta \sum_{t=1}^{2} V_{n-1}^Q(k_n, \lambda_t) \, dF(\lambda_t, \lambda) \right\}, \tag{31}$$

where $N_n = N_n(k_n, k, \lambda)$ solves (30).

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